Error analysis techniques in brief
Disclaimer: error analysis in general is quite complicated. Statistical analysis is often more complicated than I have described here. This is not a definitive guide but simply an introduction.

In many experiments you *measure* some quantities then use those measurements to *calculate* another quantity. There are different techniques for determining uncertainties (errors) for *measured* versus *calculated* values. One way to understand why we care is to think "all this crap helps determine how many sig figs we should write for each number". The number of sig figs you write matters to people who read or hear your work.

Percent *difference* is used to compare the final *measured* or *calculated* value to the *predicted* or *accepted* value. Note: percent *difference* and percent *uncertainty* (a.k.a. percent error or percent precision) are not the same thing. If *%diff* is significantly greater than *%uncert* the experimental value is <u>not</u> in good quantitative agreement with the predicted or accepted value.
If you want to express your result using confidence limits, read the lab manual section "Introduction to Uncertainty".

Determine uncertainty in a *measured* quantity (usually mass, length, time, etc)
1) Suppose I measure mass *once*. The scale reading say 12.20 g. Typically we assume the scale reading is off by at least 0.005 g. In my classes I often over estimate this uncertainty as 0.01 g just to keep the data sheets looking nice. I say $m = 12.20 \pm 0.01$ g.
   a. *Exception 1:* If you were really hard charging you would read the manual for the balance online somewhere to see if the manufacturer suggests a different error bar. For instance, some thermometers we have in the lab say the uncertainty is actually $\pm 2°C$ even though the temperature is measured to the tenths place! Talk about false advertising…
   b. *Exception 2:* Perhaps the number on the balance is fluctuating. Perhaps you had trouble starting or stopping a timer due to reaction time. Perhaps you had to hold a ruler at distance while making a measurement. Any of these may increase your uncertainty. Be realistic when you write down you uncertainty to incorporate this information.
2) Suppose I measure the mass required to balance a hockey puck *five* times at *five* different places on a board. All five measurements should be nearly identical. Determine the uncertainty using one of the following:
   a. Get the average value. In Excel use "=AVERAGE()" then highlight the values. Determine the uncertainty from the range of values. Use uncertainty is $\frac{max-min}{2}$.

Trial	m (g)	
1	12.20	
2	12.27	
3	12.32	*max*
4	12.19	
5	12.08	*min*
average	12.212	
uncertainty	0.12	

   Now think about the sig figs. Round any uncertainties to one sig fig OR to two sig figs if the first digit is 1. In this case the first digit is 1 in 0.12 so we keep two sig figs. Since the uncertainty is in the tenths and hundredths column, round the average to those columns. **Our final result from this technique is $m = 12.21 \pm 0.12$ g. This is the most <u>conservative</u> estimate of error.**
   More bullets next page…

b. Get the average value using "=AVERAGE" in Excel. Determine the uncertainty from the standard deviation using "=STDEV". Be careful! Do <u>not</u> include the average value in your standard deviation calculation or you will artificially lower your estimate for uncertainty.

Trial	m (g)	
1	12.20	
2	12.27	
3	12.32	
4	12.19	
5	12.08	
average	12.212	
stdev	0.09094	Round to 1 s.f., 2 s.f. if first digit is 1

Our final result from this technique is $m = 12.21 \pm 0.09$ g. This is the most <u>standard</u> estimate of <u>statistical</u> error. It implies 68% of all subsequent measurements are expected to fall between 12.12 g & 12.30 g. **Watch out! If statistical error is smaller than the error in reading the measurement device, read below.**

c. If all five values are identical or nearly identical, the standard estimate of statistical error sometimes gives an artificially low estimate of uncertainty. Divide the uncertainty of the balance by $\sqrt{N}$ where N is the number trials. The larger of these estimates is your uncertainty in the mass.
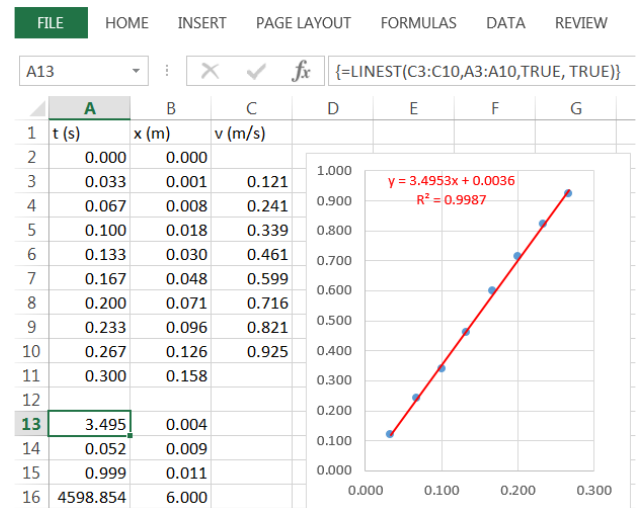
Trial	m (g)			
1	12.20			
2	12.20		0.02	Manufacturer's estimate of balance precision
3	12.19		0.01	Divided by $\sqrt{N}$
4	12.20		**0.01**	Round to 1 s.f.
5	12.20			Use **larger** of the (green and
average	12.198			**blue**) two bold error estimates
stdev	**0.00447**	Round to 1 s.f., 2 s.f. if first digit is 1		

Our final result from this technique is $m = 12.20 \pm 0.01$ g.

3) Suppose I use video capture (or an encoder) to obtain t, x, and v data. I use the slope of a vt-plot to determine an experimental value of acceleration. I read the lab manual appendices regarding the LINEST function in Excel which computes the statistical error in the slope.

My LINEST scratch work is shown below the data at right. Notice the plot is unnecessary if you use LINEST. This is why I chose not to format the vt-plot. **Final result is $a = 3.50 \pm 0.05 \frac{m}{s^2}$.**

Watch out! Don't blindly believe this number. Think about the sig figs of your raw data. These sometimes have usually high precision. The positions are off in the second sig fig. One might consider rounding this a to the tenths column instead of the hundredths column.

FILE   HOME   INSERT   PAGE LAYOUT   FORMULAS   DATA   REVIEW

A13 f_x {=LINEST(C3:C10,A3:A10,TRUE, TRUE)}

	A	B	C	D	E	F	G
1	t (s)	x (m)	v (m/s)				
2	0.000	0.000					
3	0.033	0.001	0.121				
4	0.067	0.008	0.241				
5	0.100	0.018	0.339				
6	0.133	0.030	0.461				
7	0.167	0.048	0.599				
8	0.200	0.071	0.716				
9	0.233	0.096	0.821				
10	0.267	0.126	0.925				
11	0.300	0.158					
12							
13	3.495	0.004					
14	0.052	0.009					
15	0.999	0.011					
16	4598.854	6.000					

y = 3.4953x + 0.0036
R² = 0.9987

Determine uncertainty in a *calculated* value using a *single* trial:

Now suppose I want to use the equation

$$x = \left(\frac{8\mathcal{P}}{9m}\right)^{1/2} t^{3/2}$$

where x is distance, m is mass, t is time, and $\mathcal{P}$ is power. I plan to measure x, m, and t to compute $\mathcal{P}$.
First I solve the equation for $\mathcal{P}$:

$$\mathcal{P} = \frac{9mx^2}{8t^3}$$

I read error analysis formulas from the last page of the lab manual appendices to discover

$$\frac{\delta\mathcal{P}}{\mathcal{P}} = \sqrt{\left(\frac{\delta m}{m}\right)^2 + \left(2\frac{\delta x}{x}\right)^2 + \left(3\frac{\delta t}{t}\right)^2}$$

If you are wondering where this comes from, think of each fraction as a percent error (actually called fractional error in this case, but equivalent to % error). You are essentially adding up the fractional errors (in quadrature) of each measurement to find the fractional error in your calculated value. If an exponent appears on a number, that affects the weight of that error accordingly (e.g. x^2 in the equation for $\mathcal{P}$ implies twice as much error from $\frac{\delta x}{x}$).

To get the $\pm$ number ($\delta\mathcal{P}$), don't forget to cross-multiply by $\mathcal{P}$!!!

$$\delta\mathcal{P} = \mathcal{P}\sqrt{\left(\frac{\delta m}{m}\right)^2 + \left(2\frac{\delta x}{x}\right)^2 + \left(3\frac{\delta t}{t}\right)^2}$$

Now I look my data sheet where I recorded measurements AND uncertainties.

m (kg)	δm (kg)	x (m)	δx (m)	t (s)	δt (s)	$\mathcal{P}$ (W)	$\delta\mathcal{P}$ (W)
0.500	0.001	1.00	0.01	1.34	0.08	0.23	0.04

To determine the sig figs on the final calculated result of $\mathcal{P}$ I first calculated everything in Excel, then rounded $\delta\mathcal{P}$ to a single sig fig, then rounded $\mathcal{P}$ to match. Notice, in this case all the original measurements had 3 sig figs but the final measurement only has 2!!! Most of this comes from the error in time because it is the largest fractional error and, because it is cubed in $\mathcal{P} = \frac{9mx^2}{8t^3}$, this error is tripled in the computation of $\delta\mathcal{P}$.

Determine uncertainty in a *calculated* value using a *multiple* trials and STDEV:

Suppose I tabulate several values of m_2. Perhaps these are the mass required to cause a hockey puck (m_1) to slip at various angles. I use these values of mass to compute a coefficient of friction using the formula

$$\mu_s = \frac{m_2}{m_1} - \tan \theta$$

I obtain the data table shown below. Notice I used radians for my trig function because Excel requires that. The values in the 4th column were computed using the above formula. The average was found using "=AVERAGE" and the standard deviation was found using "=STDEV".

m_1 (g)		all $\delta m's = 1$ g	
175		all $\delta \theta's = 1$ deg	
θ (deg)	θ (rad)	m_2 (g)	μ_s
0	0.00	154	0.88
15	0.26	198	0.86
30	0.52	190	0.51
45	0.79	296	0.69
60	1.05	407	0.59
75	1.31	773	0.69
		average	0.70
		stdev	0.147095

Remember to round the error calculation to 1 sig fig (or 2 sig figs if the first digit is a 1).

Our final result for μ_s is given by

$$\mu_s = 0.70 \pm 0.14$$

Note: if the statistical error is too small to be plausible, use the technique shown on the previous page for each calculated value of μ_s. For instance, we expect our data is good to about two sig figs as our measurements of angle had two sig figs. If for some reason the statistical uncertainty gave us ± 0.003 this implies three sig figs on the final answer. I would be skeptical and be concerned I am underestimating my experimental uncertainty.

At this point it would get tricky. You would be required to use the derivative-based formula on the last page of the lab manual appendices to determine the uncertainty associated with each individual value of μ_s. If this issue arises, ask your instructor for help.

Determine uncertainty in a *calculated* value using a *multiple* trials and a graph:

Now suppose I want to use the equation

$$x = \left(\frac{8\mathcal{P}}{9m}\right)^{1/2} t^{3/2}$$

where x is distance, m is mass, t is time, and $\mathcal{P}$ is power. I plan to measure x, m, and t to compute $\mathcal{P}$.

A plot of x versus $t^{3/2}$ should be linear with $slope = \left(\frac{8\mathcal{P}}{9m}\right)^{1/2}$.

I can rearrange the above equation to find

$$\mathcal{P} = \frac{9m(slope)^2}{8}$$

From work similar to the previous page I know

$$\delta\mathcal{P} = \mathcal{P}\sqrt{\left(\frac{\delta m}{m}\right)^2 + \left(2\frac{\delta slope}{slope}\right)^2}$$

First I measure the mass of my object and record the uncertainty.

m (kg)	δm (kg)
0.500	0.001

Now I use video tracking software or an encoder to record position versus time data. Notice I also record the uncertainty in each position and time measurement. I estimated these based on the blurriness of my video, from the sig figs of the measurements, or by reading the manufacturer's manual for the encoder. Notice the errors in position. As the object picked up speed it got harder to locate the position in Tracker. This explains why δx increases.

x (m)	δx (m)	t (s)	δt (s)
0.000	0.000	0.000	0.001
0.009	0.001	0.033	0.001
0.030	0.002	0.067	0.001
0.080	0.003	0.100	0.001
0.150	0.010	0.133	0.001
0.210	0.010	0.167	0.001
0.290	0.020	0.200	0.001

Now recall I needed to plot x versus $t^{3/2}$. I make two extra columns in my data sheet for $t^{3/2}$ and $\delta t^{3/2}$.

Note: $\delta t^{3/2} = \frac{3}{2}\delta t$ using the last page of the lab manual appendices.

Notice the units also change!!!

| x (m)	δx (m)	t (s)	δt (s)	$t^{3/2}$ (s$^{3/2}$)	$\delta t^{3/2}$ (s$^{3/2}$)
0.000	0.000	0.000	0.001	0.000	0.0015
0.009	0.001	0.033	0.001	0.006	0.0015
0.030	0.002	0.067	0.001	0.017	0.0015
0.080	0.003	0.100	0.001	0.032	0.0015
0.150	0.010	0.133	0.001	0.049	0.0015
0.210	0.010	0.167	0.001	0.068	0.0015
0.290	0.020	0.200	0.001	0.089	0.0015

Now I have all I need to make the plot x versus $t^{3/2}$. This is done on the next page…

When making a plot it is always stated as vertical coordinate versus horizontal coordinate (y vs x). If we plan to plot x versus $t^{3/2}$ we know x goes on the vertical axis and $t^{3/2}$ on the horizontal in this case. To make it easier to plot in Excel, I made an extra copy of the first two columns. Excel assume the leftmost column

x (m)	δx (m)	t (s)	δt (s)	$t^{3/2}$ (s$^{3/2}$)	$\delta t^{3/2}$ (s$^{3/2}$)	x (m)	δx (m)
0.000	0.000	0.000	0.001	0.000	0.0015	0.000	0.000
0.009	0.001	0.033	0.001	0.006	0.0015	0.009	0.001
0.030	0.002	0.067	0.001	0.017	0.0015	0.030	0.002
0.080	0.003	0.100	0.001	0.032	0.0015	0.080	0.003
0.150	0.010	0.133	0.001	0.049	0.0015	0.150	0.010
0.210	0.010	0.167	0.001	0.068	0.0015	0.210	0.010
0.290	0.020	0.200	0.001	0.089	0.0015	0.290	0.020
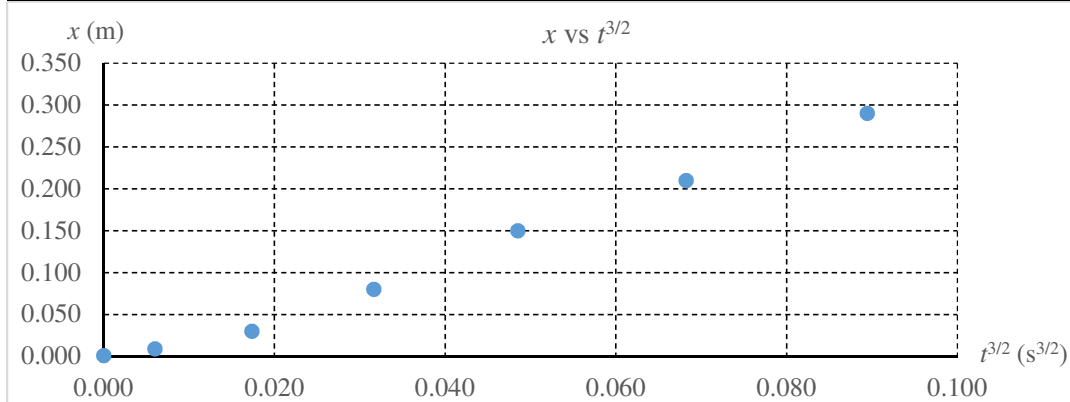
To get the error bars, recall we put x on the vertical axis and $t^{3/2}$ on the horizontal axis. Therefore the horizontal error bars use the 6th column and the vertical error bars use the 8th (or 2nd) column. To add in the error bars I "added a chart element" and clicked on error bars. Once the error bars appeared I right clicked on a vertical error bar, clicked "format error bar", clicked "custom", hit the button to "specify value", then highlighted the numbers only in the 8th column. I did something similar for the horizontal error bars. Once both error bars were correct, I removed the dots (marker, marker options, no marker). Finally, I added a trendline showing both the equation and R^2 value on the chart.
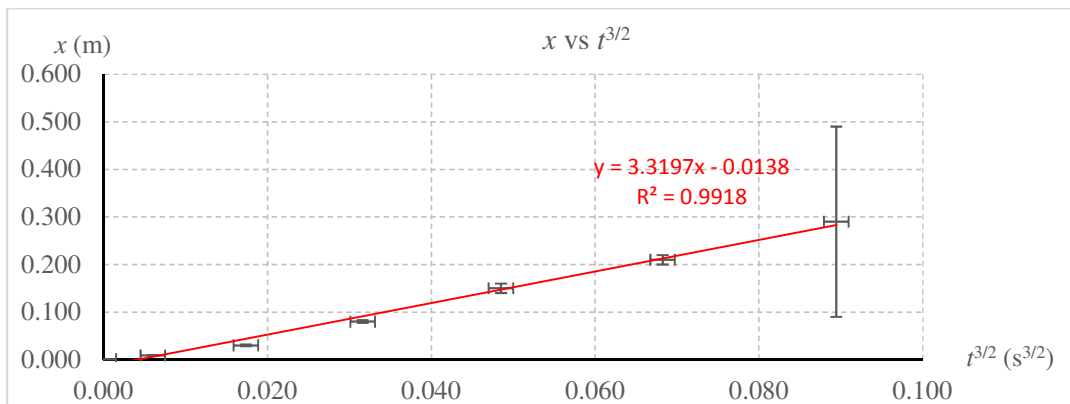
Remember what we started with.

I want to use the equation $x = \left(\frac{8\mathcal{P}}{9m}\right)^{1/2} t^{3/2}$. I plan to measure x, m, and t to compute $\mathcal{P}$. A plot of x versus $t^{3/2}$

should be linear with $slope = \left(\frac{8\mathcal{P}}{9m}\right)^{1/2}$. I can rearrange the above equation to find

$$\mathcal{P} = \frac{9m(slope)^2}{8}$$

$$\delta\mathcal{P} = \mathcal{P}\sqrt{\left(\frac{\delta m}{m}\right)^2 + \left(2\frac{\delta slope}{slope}\right)^2}$$

To get the error in the slope I can use the LINEST function as described in the lab manual appendices. A screen shot of my data is shown at right. I included my scratch work for the LINEST function. From the LINEST work I determine $slope = 3.32$ and $\delta slope = 0.14$. I chose to round to the hundredths columns because the error in the slope should be rounded to 1 sig fig (or 2 sig figs if the first error digit is a 1). This told me the appropriate column for rounding the slope. The units of $slope$ (and $\delta slope$) are units of rise (vertical axis) over units of run (horizontal axis). Therefore

$$slope = 3.32 \pm 0.14\frac{m}{s^{3/2}}$$

fx | {=LINEST(Z10:Z16,X10:X16, TRUE, TRUE)}

X	Y	Z	AA
$t^{3/2}$ (s$^{3/2}$)	$\delta t^{3/2}$ (s$^{3/2}$)	x (m)	δx (m)
0.000	0.0015	0.001	0.000
0.006	0.0015	0.009	0.001
0.017	0.0015	0.030	0.002
0.032	0.0015	0.080	0.003
0.049	0.0015	0.150	0.010
0.068	0.0015	0.210	0.010
0.089	0.0015	0.290	0.200

LINEST scratchwork

3.3197015	-0.0138497
0.1353235	0.0065394
0.99176	0.0109969
601.7993	5

Now I can compute the final result for $\mathcal{P}$ and determine the associated uncertainty $\delta\mathcal{P}$.

$$\mathcal{P} = 6.2 \pm 0.5 \text{ W}$$

Said another way, we can describe the % error using

$$\% \text{ uncertainty in calculated value of power} = \frac{\delta\mathcal{P}}{\mathcal{P}} \times 100\% = 8\%$$

Getting a value for power is easy. Figuring out the %uncertainty (a.k.a. %error or %precision) is no joke. Still, I hope you agree it feels a lot more meaningful to understand not only how to get a number but also to get a feel for the *quality* of the number.