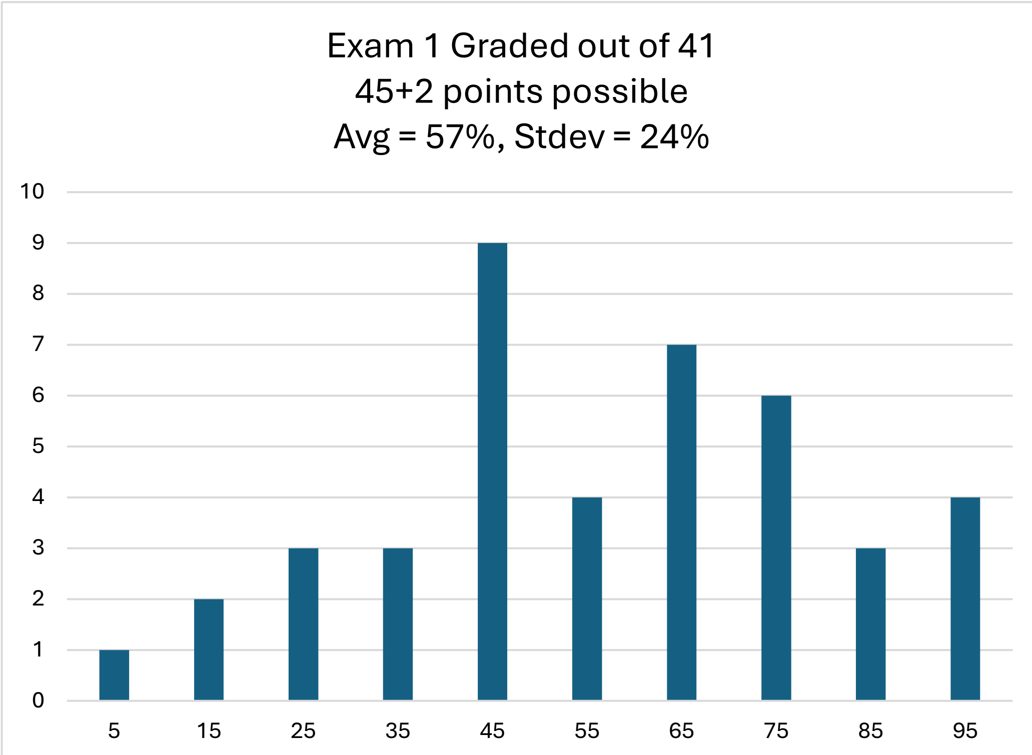
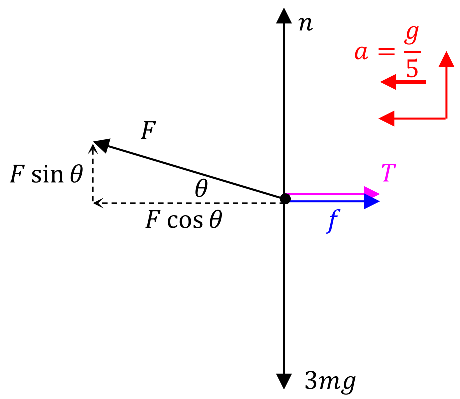
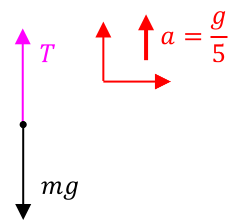
161fa25t2a
Distribution on this page.
Solutions begin on the next page.



1) Start with free body diagrams and force equations. *Most people who tried the system FBD screwed it up…*





Block 3*m*	**Block *m***
ΣF_x: $F\cos\theta - T - f = 3ma$	ΣF_x: $T - mg = ma$ A common mistake was assuming $T = mg$.
Because $a_y = 0$ I often use "the ups" = "the downs" ΣF_y: $n + F\sin\theta = 3mg$	

Stacking the equations and adding them is often the easiest way to eliminate internal forces.

$$F\cos\theta - T - f = 3ma$$

$$+ \qquad T - mg = ma$$

$$F\cos\theta - f - mg = 4ma$$

We know the blocks are sliding. Use $f = \mu_k n$. Determine n from the vertical force equation for block $3m$.

$$n = 3mg - F\sin\theta$$

$$f = \mu_k(3mg - F\sin\theta)$$

Plug in this for friction. Plug in $a = \frac{g}{5}$. Solve for F (and simplify). Watch out for those minus signs!

$$F\cos\theta - \mu_k(3mg - F\sin\theta) - mg = 4m\frac{g}{5}$$

$$F\cos\theta - 3\mu_k mg + \mu_k F\sin\theta = mg + \frac{4}{5}mg$$

$$F(\cos\theta + \mu_k\sin\theta) = \frac{9}{5}mg + 3\mu_k mg$$

$$F = mg\frac{\frac{9}{5} + 3\mu_k}{\cos\theta + \mu_k\sin\theta}$$

Plug in numbers for θ & μ_k.

$$F = mg\frac{1.8 + 3(0.333)}{\cos(10.00) + 0.333\sin(10.00°)}$$

$$\boldsymbol{F = 2.68mg}$$

Check: verify $n > 0$.

$$n = 3mg - \boldsymbol{2.68mg}\sin(10.00°) = 2.53mg > 0$$

2a) You can consider system diagrams for systems m_1 only, $m_2 + m_1$, and finally $(m_3 + m_5) + m_2 + m_1$.

- For the m_1 only system, the horizontal force equation gives $T_1 = m_1 a$.
- For the $m_2 + m_1$ system, the horizontal force equation gives $T_2 = (m_2 + m_1)a$.
- For the $(m_3 + m_5) + m_2 + m_1$ system, the horizontal force equation gives $T_3 = (m_1 + m_2 + m_3 + m_5)a$.

$$\boldsymbol{T_3 > T_2 > T_1}$$

2b) Consider an FBD of block m_2. The horizontal force equation gives

$$T_2 - T_1 = m_2 a$$

We expect $\boldsymbol{T_2 = T_1}$ whenever $\boldsymbol{a = 0}$. I used the approximate sign in the question statement because they would never be *exactly* equal in the real world due to a tiny amount of friction or drag.

2c) **NEVER true.** According to Newton's third law, the normal force exerted by m_3 on m_5 is equal and opposite to the normal force exerted by m_5 on m_3. This is true for any acceleration!

2d) **Positive.** The normal force does no NET work on the system…but it supports the motion of m_5.

2e) **Zero.** The force of gravity is perpendicular to displacement.

3) Kinetic energy is given by $K = \frac{1}{2}mv^2$. Notice this is always positive. That eliminates most of the graphs.
Think: the block should experience constant acceleration.
If we assume up the ramp is the positive direction

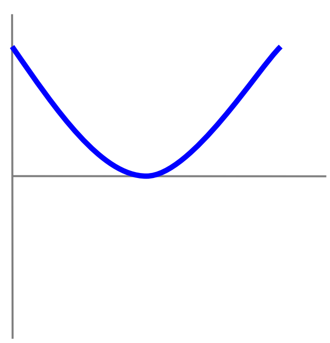
$$v(t) = v_i - at$$

Plug this into the kinetic energy equation.

$$K(t) = \frac{1}{2}m(v_i - at)^2$$

$$K(t) = \frac{1}{2}mv_i^2 - mv_i at + \frac{1}{2}ma^2t^2$$

Notice this is a parabola, always positive, with initial value $K_i = \frac{1}{2}mv_i^2$.
Note: if you chose a different coordinate system you should still get the same result!
Notice kinetic energy is never negative & is zero when the block is at the top of the ramp (at half of total time).
The only plausible sketch is



4a) Consider the figures shown at right.
Force equations give

$$a_{tan} = g = 9.8\,\frac{\text{m}}{\text{s}^2}$$

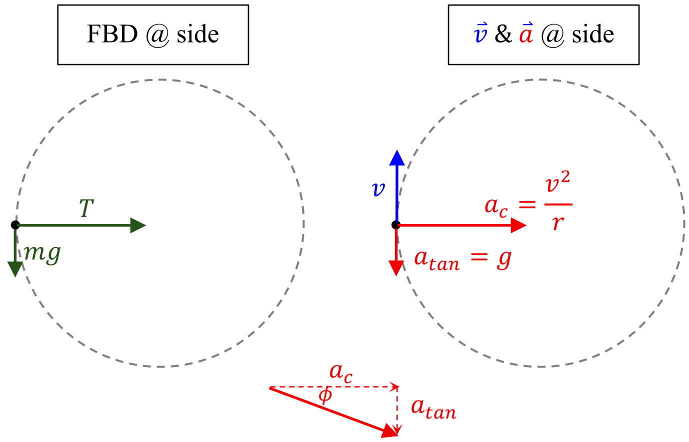
$$a_c = \frac{v^2}{r} = 35.52\,\frac{\text{m}}{\text{s}^2}$$

Magnitude of total acceleration is

$$a_{total} = \sqrt{a_c^2 + a_{tan}^2} = \mathbf{36.8\,\frac{m}{s^2}}$$

I found my angle using

$$\phi = \tan^{-1}\left(\frac{a_{tan}}{a_c}\right) = \mathbf{15.42°}$$



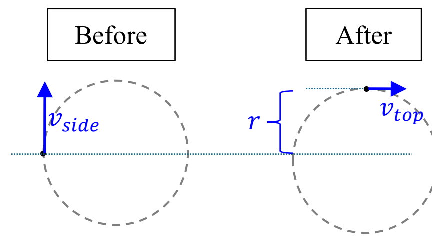
4b) To determine speed at the top of the loop, it is easiest to do an energy problem!
Tension is always directed perpendicular to displacement (and thus does no work).
Air resistance is negligible.

$$mgh_i + \frac{1}{2}mv_i^2 + W_{\substack{ext\\non-con}} = mgh_f + \frac{1}{2}mv_f^2$$

$$0 + \frac{1}{2}mv_{side}^2 + 0 = mgr + \frac{1}{2}mv_{top}^2$$

$$v_{top} = \sqrt{v_{side}^2 - 2gr}$$

$$\boldsymbol{v_{top} = 2.972\,\frac{m}{s}}$$



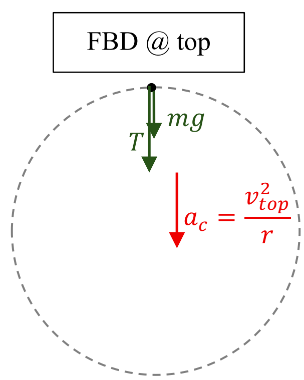
Notice this is larger than the minimum speed required for circular motion at the top given by $v_{min} = \sqrt{rg} \approx 2.33\,\frac{\text{m}}{\text{s}}$.

4c) The FBD at the top generates the force equation

$$T + mg = \frac{mv_{top}^2}{r}$$

$$T = \frac{mv_{top}^2}{r} - mg$$

$$\boldsymbol{T = 1.358\ \text{N}}$$

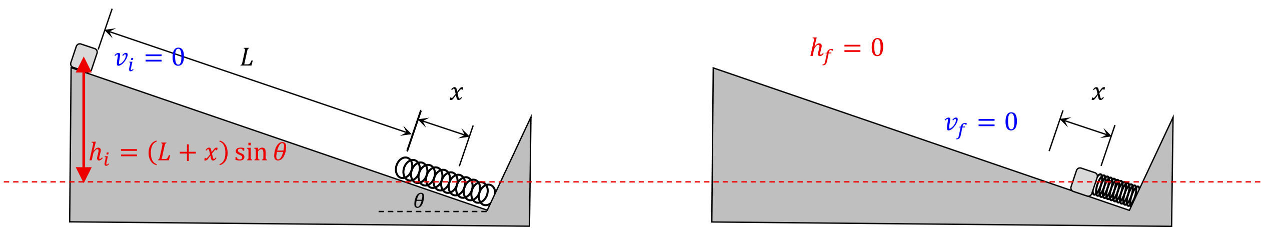


This seems reasonable…we should get a small but positive value for T if speed at the top is larger than v_{min}.

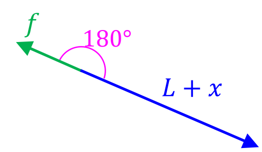
Did you remember to convert mass to kg?

5a) The block and spring exert equal magnitude forces on each other according to Newton's 3rd law.

5b) Draw before and after pictures. Think: at the instant the block reverse direction the spring is at max compression and the block has zero velocity.



An FBD of the block while sliding gives $n = mg\cos\theta$ giving kinetic friction magnitude $f = \mu_k mg\cos\theta$.
Recall, in this problem we were told to use $\mu_k = \mu$.
The friction does work $W = -\mu mg(L + x)\cos\theta$.
The minus sign here comes from the fact friction is angled 180° from displacement!



$$\frac{1}{2}mv_i^2 + mgh_i + \frac{1}{2}kx_i^2 + W_{\substack{ext\\non-con}} = \frac{1}{2}mv_f^2 + mgh_f + \frac{1}{2}kx_f^2$$

$$0 + mgh_i + 0 + W_{\substack{ext\\non-con}} = 0 + 0 + \frac{1}{2}kx_f^2$$

$$mg(L + x)\sin\theta - \mu mg(L + x)\cos\theta = \frac{1}{2}kx^2$$

$$mg(L + x)[\sin\theta - \mu\cos\theta] = \frac{1}{2}kx^2$$

$$\boldsymbol{m = \frac{kx^2}{2g(L + x)[\sin\theta - \mu\cos\theta]}}$$

Unit check:

$$\frac{\frac{\text{N}}{\text{m}} \cdot \text{m}^2}{\frac{\text{m}}{\text{s}^2} \cdot \text{m}} = \frac{\text{kg} \cdot \frac{\text{m}}{\text{s}^2}}{\text{m}} \cdot \text{m}^2 \cdot \frac{\text{s}^2}{\text{m}^2} = \text{kg}$$

For fun, contemplate various possible scenario modifications (e.g. longer ramp, stiffer spring, larger angle) and use those thought experiments to see if this equation makes sense.

For example, if the spring compressed farther, we know x would be bigger; our equation for m indicates mass would have to increase.

Some of the other thought experiments are actually pretty complicated for this problem. For instance, do we require more or less mass for a longer ramp (larger L)? At first this seems counterintuitive…but it does check out. Think: if compression distance x is unchanged, a longer ramp (larger L) gives a taller h_i which requires less mass to get the same initial gravitational potential energy! Friction complicates things as well. This would be a good case to analyze using a simulation…

6a) On plots of U vs x we know

$$F_x = -slope$$

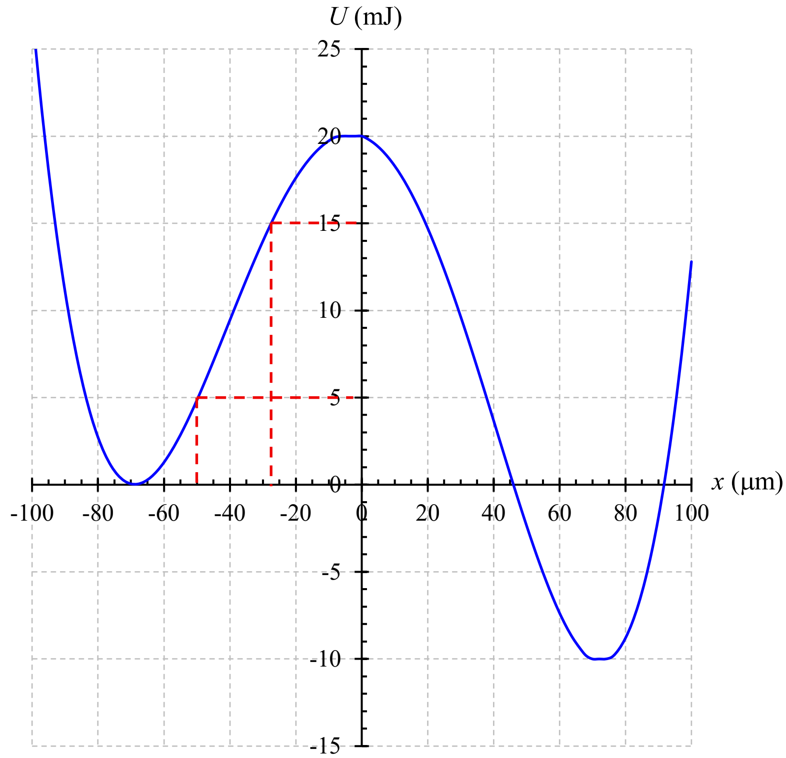
The slope at $x = -40.0$ μm is positive.
Therefore $F_x < 0$.
This implies ***force to the left***.

6b) Acceleration *magnitude* relates to force using

$$|a_x| = \frac{|F_x|}{m}$$

$$|a_x| \approx \frac{|-slope|}{m}$$

$$|a_x| \approx \frac{\frac{rise}{run}}{m}$$



WATCH OUT FOR UNIT PREFIXES!

$$|a_x| \approx \frac{\frac{10.0 \text{ mJ}}{22.5 \text{ μm}}}{3.33 \text{ pg}}$$

$$|a_x| \approx \frac{\frac{10.0 \times 10^{-3} \text{ J}}{22.5 \times 10^{-6} \text{ m}}}{3.33 \times 10^{-12} \text{ g}}$$

DOUBLE WATCH OUT! We need mass in kg since $1 \text{ J} = 1 \text{ N} \cdot \text{m} = 1 \text{ kg} \cdot \frac{\text{m}^2}{\text{s}^2}$.

$$|a_x| \approx \frac{\frac{10.0 \times 10^{-3} \text{ kg} \cdot \frac{\text{m}^2}{\text{s}^2}}{22.5 \times 10^{-6} \text{ m}}}{3.33 \times 10^{-15} \text{ kg}}$$

$$\boldsymbol{|a_x| \approx 1.34 \times 10^{17} \frac{\text{m}}{\text{s}^2}}$$

Side note: there is probably no point in keeping more than two sig figs since we are reading off the plot.
Hopefully you got a result within about 10% of my value to get full credit.
If you are worried, this doesn't violate special relativity.
Accelerations in the real world can be enormous over short time intervals for low mass objects.
Speed is limited by the speed of light.

Last note: when you are asked to put something in scientific notation it is expected you will use standard units. The standard units of acceleration are $\frac{\text{m}}{\text{s}^2}$. Part of your job of being a scientist is writing your results in a standard form to make life easier for anyone reading your work. You lost points if you failed to clean up all the prefixes correctly.

7) Don't overthink it. Average power is given by

$$\mathcal{P}_{avg} = \frac{\Delta E}{\Delta t}$$

In this case, the car's kinetic energy is changing.

$$\mathcal{P}_{avg} = \frac{\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2}{\Delta t}$$

$$\mathcal{P}_{avg} = m\frac{v_f^2 - v_i^2}{2\Delta t}$$

$$\mathcal{P}_{avg} = (1250\text{ kg})\frac{\left(30.0\frac{\text{m}}{\text{s}}\right)^2 - 0}{2(4.00\text{ s})}$$

$$\mathcal{P}_{avg} = 140625\ \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2\cdot\text{s}}$$

$$\mathcal{P}_{avg} = 140625\ \frac{\text{J}}{\text{s}}$$

$$\boldsymbol{\mathcal{P}_{avg} = 140.6\text{ kW}}$$

Notice this is *not* the same thing as $\frac{\frac{1}{2}mv_{avg}^2}{\Delta t}$…

Verifying this fact: I used kinematics to find total distance traveled is 77.5 m in 4.00 s giving $v_{avg} = 19.\underline{3}75\,\frac{\text{m}}{\text{s}}$.

$$\frac{\frac{1}{2}mv_{avg}^2}{\Delta t} = 58.7\text{ kW} \neq 140.6\text{ kW}$$

Notice you should not determine power for the two time intervals then average them!
Think: in general this is not a good idea, especially when the two time intervals are not identical.
If something ran at 1.00 kW for 1.00 s then at 2.00 kW for 100.0 s the average power output should be pretty darn close to 2.00 kW (not 1.50 kW).

8a) Remember, we can ignore constants (e.g., -1) during the unit check. Remember those brackets on $[k]$!

$$[k] = [F] \cdot [x]^3 = \mathbf{N \cdot m^3} = \mathbf{J \cdot m^2} = \mathbf{kg \cdot \frac{m^4}{s^2}}$$

Any of these bolded forms is fine. Since we are going to compute energy, it made sense to me to use the middle, highlighted result.

8b) Recall

$$\Delta U = -Work$$

$$\Delta U = -\int_i^f F_x \, dx$$

$$\Delta U = -\int_i^f -\frac{k}{x^3} \, dx$$

$$\Delta U = \int_i^f kx^{-3} \, dx$$

$$\Delta U = \left[\frac{kx^{-2}}{-2}\right]_d^{3d}$$

Get rid of the minus sign by flipping the order of the limits.

$$\Delta U = \left[\frac{kx^{-2}}{2}\right]_{3d}^{d}$$

$$\Delta U = \left[\frac{k}{2x^2}\right]_{3d}^{d}$$

$$\Delta U = \frac{k}{2d^2} - \frac{k}{18d^2}$$

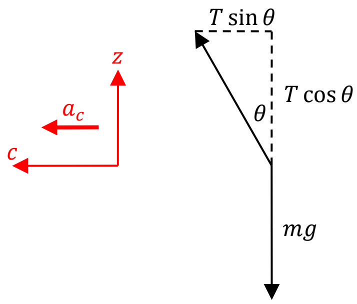
$$\Delta U = \frac{9k}{18d^2} - \frac{k}{18d^2}$$

$$\Delta U = \frac{8k}{18d^2}$$

$$\boldsymbol{\Delta U = \frac{4k}{9d^2} \approx 0.444 \frac{k}{d^2}}$$

Notice units are NOT included on algebraic answers.
Think! The force is directed to the left for positive values of x.
As the particle moves to the right, the force does negative work.
Because we know $\Delta U = -Work$ we should expect a positive change in potential energy.
Units also check out: $\frac{\mathrm{J \cdot m^2}}{\mathrm{m^2}} = \mathrm{J}$.
Notice that middle form of the units for k was pretty handy.

9a) The FBD and force equations.



ΣF_c: $T\sin\theta = ma_c = mr\omega^2$	ΣF_z: $T\cos\theta = mg$

Think: know $a_c = \frac{v^2}{r}$ and $v = r\omega$. Subbing in for v gives $a_c = r\omega^2$ (which is also on your equation sheet). Since the problem statement gave us ω as a given, it makes sense to use $a_c = r\omega^2$ in our equations.

Use a ratio on the force equations.

$$\frac{T\sin\theta}{T\cos\theta} = \frac{mr\omega^2}{mg}$$

$$\tan\theta = \frac{r\omega^2}{g}$$

Think: we are trying to find L, the length of the string.
In this problem, L relates to radius using $r = L\sin\theta$.

$$\tan\theta = \frac{(L\sin\theta)\omega^2}{g}$$

$$\boldsymbol{L = \frac{g\tan\theta}{\omega^2\sin\theta}}$$

I'll accept this answer but, strictly speaking, it is good etiquette to simplify as shown below.

$$L = \frac{g\frac{\sin\theta}{\cos\theta}}{\omega^2\sin\theta}$$

$$\boldsymbol{L = \frac{g}{\omega^2\cos\theta}}$$

9b) I find it helps to first write out a sentence for the action force (the weight force acting on the ball) before writing the reaction force.

Action: The earth exerts a gravitational force on the ball directed downwards .

Reaction: The ball exerts a gravitational force on the earth directed upwards .

Extra Credit 1 – Use the given information to determine $v_1(t)$ for the first time interval.

$$v_1(t) = v_{1i} + a_1 t$$

We know $v_{1i} = 0$:

$$v_1(t) = a_1 t$$

We also know $v_1(t) = 20.0\,\frac{\text{m}}{\text{s}}$ at time $t = 1.500$ s. One finds $a_1 = 13.33\,\frac{\text{m}}{\text{s}^2}$.

Kinetic energy as a function of time is thus:

$$K_1(t) = \frac{1}{2} m [v_1(t)]^2$$

$$K_1(t) = \frac{1}{2} m [a_1 t]^2$$

$$K_1(t) = \frac{1}{2} m a_1^2 t^2$$

Instantaneous power is:

$$\mathcal{P}_{inst\ 1} = \frac{dE_1}{dt}$$

Here the power output is directly linked to the change in *kinetic* energy. Note: if we had drag or gravitational forces present, it would complicate our discussion. Ask me in person if you are curious how…

$$\mathcal{P}_{inst\ 1} = \frac{d}{dt}\left(\frac{1}{2} m a_1^2 t^2\right)$$

$$\mathcal{P}_{inst\ 1} = m a_1^2 t$$

$$\boldsymbol{\mathcal{P}_{inst\ 1} = \left(222\,\frac{\textbf{kW}}{\textbf{s}}\right) t}$$

Notice: this expression has both numerical constants and algebraic parameters. Include units on the numerical constants but not on the algebraic parameters. The units required for the algebraic part become obvious upon inspection.

Extra Credit 2 – The process is the same but the math is complicated due to this time interval starting at 1.500 s instead of $t = 0$!

If time *did* start at zero, we go from 20 to 30 in 2.50 seconds giving $a_2 = 4.00\,\frac{\text{m}}{\text{s}^2}$.

Now use a time shift ($t \rightarrow t - 1.500$ s)to write

$$v_2(t) = v_{2i} + a_2(t - 1.500\text{ s})$$

$$v_2(t) = 20.0\,\frac{\text{m}}{\text{s}} + 4.00\,\frac{\text{m}}{\text{s}^2}(t - 1.500\text{ s})$$

Now proceed as before:

$$K_2(t) = \frac{1}{2} m [v_2(t)]^2 = \frac{1}{2} m \left[20.0\,\frac{\text{m}}{\text{s}} + 4.00\,\frac{\text{m}}{\text{s}^2}(t - 1.500\text{ s})\right]^2$$

$$\mathcal{P}_{inst\ 2} = \frac{d}{dt}\left\{\frac{1}{2} m \left[20.0\,\frac{\text{m}}{\text{s}} + 4.00\,\frac{\text{m}}{\text{s}^2}(t - 1.500\text{ s})\right]^2\right\}$$

$$\mathcal{P}_{inst\ 2} = \left(4.00\,\frac{\text{m}}{\text{s}^2}\right) m \left[20.0\,\frac{\text{m}}{\text{s}} + 4.00\,\frac{\text{m}}{\text{s}^2}(t - 1.500\text{ s})\right]$$

$$\mathcal{P}_{inst\ 2} = \left(80.0\,\frac{\text{m}^2}{\text{s}^3}\right) m + \left(16.00\,\frac{\text{m}^2}{\text{s}^4}\right) m t - \left(24.0\,\frac{\text{m}^2}{\text{s}^3}\right) m$$

$$\boldsymbol{\mathcal{P}_{inst\ 2} = \left(20.0\,\frac{\textbf{kW}}{\textbf{s}}\right) t - 70.0\ \textbf{kW}}$$

Think: we need much less force when we have a much smaller acceleration. Be aware this expression is only valid for times during stage 2 ($t = 1.500$ s $\rightarrow$ 4.00 s). Plugging in $t = 1.500$ s gives $\boldsymbol{\mathcal{P}_{inst\ 2\ initial}} = \mathbf{10\underline{0}\ kW.}$