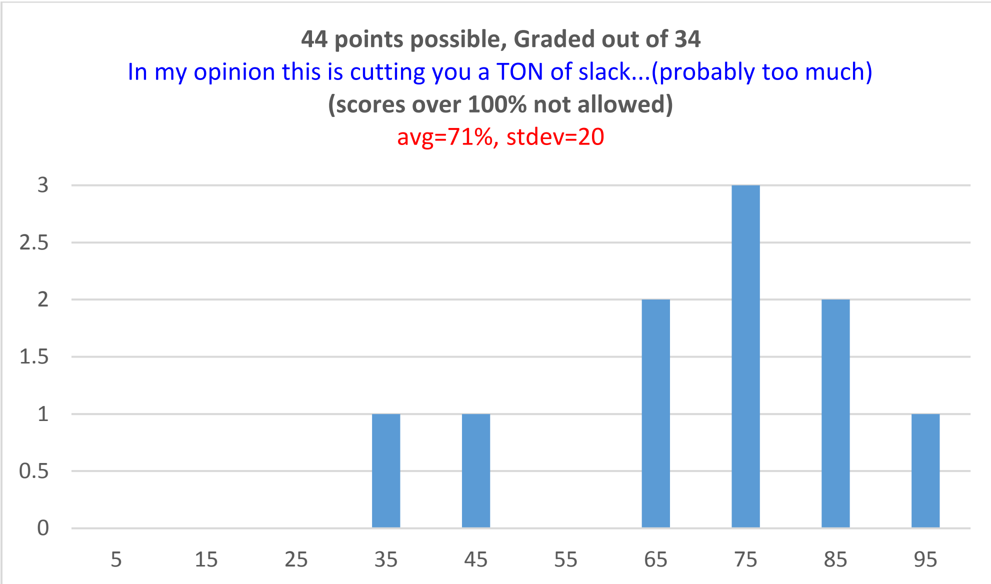
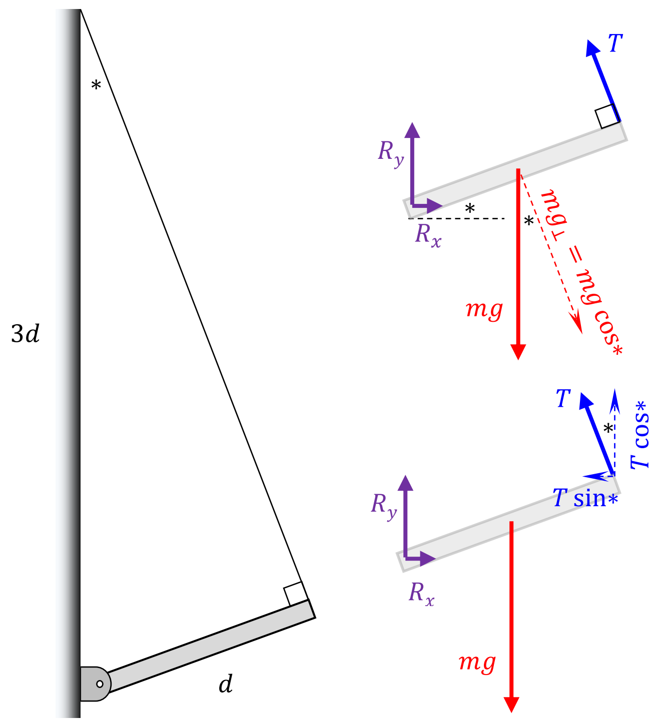
161sp22t3aSoln

Distribution on this page.

Solutions begin on the next page.



1a) $* = \sin^{-1}\left(\frac{d}{3d}\right) = 19.47°$



1b) Torques about the pivot implies the magnitude of torque from the tension must balance the magnitude of torque from the rod's weight. Using the *upper* FBD in the figure at right I found

$$LT = \frac{L}{2} mg \cos *$$

From there I used the angle in part 1 to find

$$\boldsymbol{T = 0.47\underline{1}4mg}$$

I will keep the extra digit since I know I need to use this number in subsequent calculations.

1c) Now I split up all forces according to a standard coordinate system (*lower* FBD shown at right).

Forces in the horizontal direction give

$$R_x = T \sin *$$

Forces in the vertical direction give

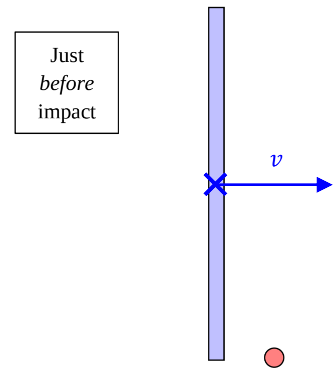
$$R_y + T\cos * = mg$$

After doing a bit of math I found

$$\boldsymbol{\vec{R} = (0.1571\hat{\imath} + 0.556\hat{\jmath})mg}$$

2a) Watch out…with rod length $2d$ the rod center of mass is at d…not $d/2$

$$y = \frac{3m(d) + 0}{3m + m} = \mathbf{\frac{3}{4}d = 0.750d}$$



2b) Must use parallel axis theorem for rod…
The system rotates about the combined center of mass (not the rod's center of mass).

$$I = I_{rod} + I_{p.m.}$$

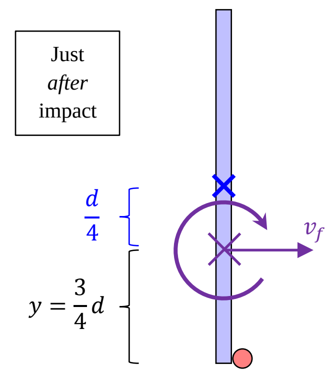
$$I = \left(I_{\substack{rod\\cm}} + m_{rod}\left(\frac{d}{4}\right)^2\right) + m_{p.m.}\left(\frac{3}{4}d\right)^2$$

$$I = \left(\frac{1}{12}(3m)(2d)^2 + (3m)\left(\frac{d}{4}\right)^2\right) + m\left(\frac{3}{4}d\right)^2$$

$$I = \left(md^2 + \frac{3}{16}md^2\right) + \frac{9}{16}md^2$$

$$\boldsymbol{I = \frac{28}{16}md^2 = \frac{7}{4}md^2}$$

$$\boldsymbol{I = 1.750md^2}$$



2c) Straight up linear momentum…perfectly inelastic collision.

$$(3m)v + 0 = (3m + m)v_f$$

$$\boldsymbol{v_f = \frac{3}{4}v}$$

2d) Angular momentum problem.
As rod moves towards point mass in straight line it has angular momentum about system center of mass.

$$L_{rod\ initial} + L_{p.m.\ initial} = L_{system\ final}$$

$$m_{rod}vr_{\perp} + 0 = I\omega_f$$

$$(3m)v\left(\frac{d}{4}\right) = \left(\frac{7}{4}md^2\right)\omega_f$$

$$\omega_f = \frac{\frac{3}{4}mvd}{\frac{7}{4}md^2}$$

$$\boldsymbol{\omega_f = \frac{3v}{7d}}$$

$$\boldsymbol{\omega_f = 0.429\frac{v}{d}}$$

3a) By Newton's 3rd law, both vehicles exert equal magnitude forces on each other regardless of acceleration!

3b) Since the force magnitudes are identical, we expect the lighter mass (in this case the car) to experience more acceleration during the collision. You might wonder about other forces acting on the vehicles. For each vehicle the normal force will offset gravitational forces. True, there might be some sliding of the wheels relative to the pavement during the impact. However, we were told in the problem statement this effect is negligible.

3c) We expect the same magnitude of change in momentum. We know force on an object is proportional to momentum change for that same object. Since the forces are identical, the momentum change is identical.

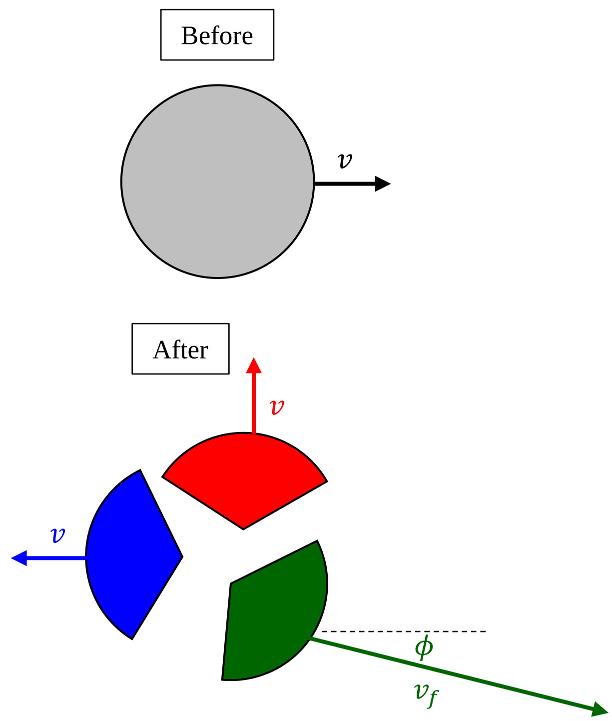
4) I redrew the before and after pictures.
Conservation of linear momentum applies.
Note: a bomb explodes so quickly the external force of gravity can be considered as negligible during an explosion.

I assumed the third fragment should probably go to the right and down based on the specifics of this problem.
Note: you could assume any direction you like for $\vec{v}_f$ since you could interpret minus signs of your final answer to get the same result as me….

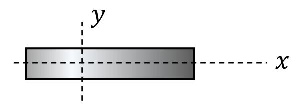
Horizontal conservation of momentum	*Vertical* conservation of momentum
$3mv = -mv + mv_{fx}$	$0 = mv - mv_{fy}$

From there I found $v_{fx} = 4v$ & $v_{fy} = v$.

The final speed and direction are $v_f = 4.12v$ & $\phi = 14.04°$.
The figure at right shows the approximate size & direction.



A rod has non-uniform mass density given by $\lambda = kx^4$.
The *left* end of the rod is distance d to the left of the origin.
The rod has length $3d$.



5a) The center of mass calculation is

$$x_{cm} = \frac{\int_{-d}^{2d} x\,(kx^4)\,dx}{\int_{-d}^{2d}(kx^4)\,dx}$$

$$x_{cm} = \frac{\left[\frac{kx^6}{6}\right]_{-d}^{2d}}{\left[\frac{kx^5}{5}\right]_{-d}^{2d}}$$

$$x_{cm} = \frac{5}{6}\cdot\frac{(2d)^6-(-d)^6}{(2d)^5-(-d)^5}$$

$$x_{cm} = \frac{5}{6}d\cdot\frac{64-1}{32+1}$$

$$\boldsymbol{x_{cm} = \frac{35}{22}d \approx 1.591d}$$

5b) The moment of inertia integral is

$$I_{yy} = \int_{-d}^{2d} x^2\,(kx^4)\,dx$$

$$I_{yy} = \left[\frac{kx^7}{7}\right]_{-d}^{2d}$$

$$I_{yy} = \frac{k}{7}[(2d)^7-(-d)^7]$$

$$\boldsymbol{I_{yy} = \frac{129}{7}kd^7 \approx 18.43kd^7}$$

5c) Use the parallel axis theorem.

$$I_{yy} = I_{cm} + m(dist\ to\ c.m.)^2$$

$$I_{cm} = I_{yy} - m(dist\ to\ c.m.)^2$$

Notice mass is given by the denominator of the center of mass integral!

$$m = total\ mass = \int dm = \int_{-d}^{2d}(kx^4)\,dx = \frac{33}{5}kd^5$$

Plugging in the pieces gives

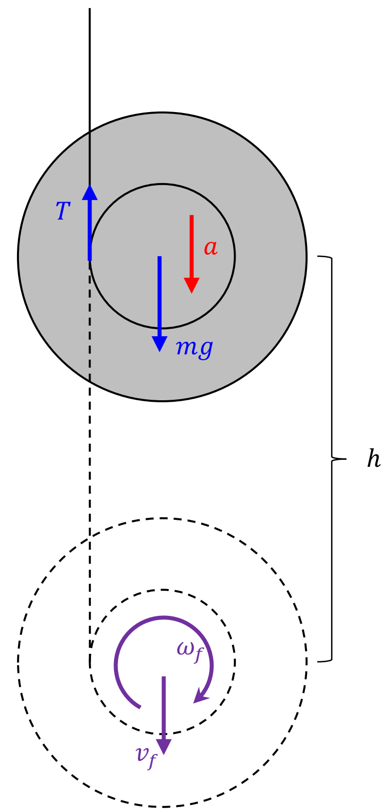
$$I_{cm} = \frac{129}{7}kd^7 - \left(\frac{33}{5}kd^5\right)\left(\frac{35}{22}d\right)^2$$

$$\boldsymbol{I_{cm} \approx 1.724kd^7}$$

5d) If rotating about the z-axis instead, the mass is still distributed the same amount from the axis!

$$\boldsymbol{I_{zz} = I_{yy}}$$

6a) The disk is not in freefall. The upwards tension causes $a < g$.



6b) We are told the disk has mass m and radius $2R$.

$$I_{cm} = \frac{1}{2}m(2R)^2 = 2mR^2$$

Doing sum of torques using the instantaneous pivot gives

$$Rmg\sin(90°) = (I_{cm} + mR^2)\alpha$$
$$Rmg = (2mR^2 + mR^2)\alpha$$
$$Rmg = (3mR^2)\alpha$$
$$g = 3R\alpha$$

Because the string unravels without slipping about axle of radius R

$$a = R\alpha$$

From there one finds

$$\boldsymbol{a = \frac{g}{3}}$$

Alternate style doing sum of torques using the center of mass pivot:
Sum of torques using the center of mass pivot gives

$$RT sin(90°) = I_{cm}\alpha$$

Vertical force equation gives

$$mg - T = ma$$

Using these two equations along with $I_{cm} = 2mR^2$ & $a = R\alpha$ gives the same result.

6c) You could use kinematics (since acceleration is constant).
Alternate style using energy is on the next page…

$$v_f^2 = v_i^2 + 2ah$$
$$v_f^2 = 0 + 2\left(\frac{g}{3}\right)h$$
$$v_f = \sqrt{\frac{2gh}{3}}$$

Now convert this to angular speed using the appropriate radius (we assume the string unravels without slipping). In this case, the string connects at radius R giving

$$\omega_f = \frac{v_f}{R}$$
$$\boldsymbol{\omega_f = \sqrt{\frac{2gh}{3R^2}}}$$

Alternate style using energy:

$$mgh = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2$$

In this case the string connects distance R from the center of mass (implying $v_f = R\omega_f$).

$$mgh = \frac{1}{2}(mR^2 + I)\omega_f^2$$

$$mgh = \frac{1}{2}(mR^2 + 2mR^2)\omega_f^2$$

From there do a bit of algebra to get the same result:

$$\boldsymbol{\omega_f = \sqrt{\frac{2gh}{3R^2}}}$$

6d) Angular momentum is conserved in the universe. As the disk unravels, it gains angular momentum in the $-\hat{k}$ direction. The person or object holding the string as it unravels is somehow coupled to the entire earth. As a result, the entire earth is given the same magnitude change of momentum in the $+\hat{k}$ direction. Because the earth's moment of inertia is so tremendous, the change in rotation rate of the earth is impossible to notice. It only *appears* angular momentum isn't conserved…

Alternatively, you could mention angular momentum is only conserved when external torque is zero.
In this case there are external forces causing torque on the disc.
Specifically, if using the instantaneous pivot, gravity caused by the earth causes the external torque.
If using the center of mass pivot, tension from the string causes the external torque.

7a) Watch out for 2π in the basement! Be sure you punch it in correctly. The conversion is

$$8.00\frac{\text{rad}}{\text{s}}\times\frac{1\text{ rev}}{2\pi\text{ rad}}\times\frac{60\text{ s}}{1\text{ min}}=\mathbf{76.4\ RPM}$$

As a check, we know conversion between $\frac{\text{rad}}{\text{s}}$ & RPM involve a factor of *approximately* 10.

7b) Angular acceleration is slope of the plot of ω vs t.
Notice the slope is constant over the entire plot in this case.

$$\boldsymbol{\alpha}=\mathbf{-2.00\frac{rad}{s^2}}$$

Here, since I didn't specify magnitude in the question, I am looking for the minus sign.
It is probably a bit dubious to expect 3 sig figs from reading a plot…but in this case things line up very well.

7c) Centripetal acceleration *magnitude* is given by

$$a_c=r\omega^2$$

Get $\omega=-1.00\frac{\text{rad}}{\text{s}}$ from the plot at $t=4.5$ s.

$$a_c=(0.375\text{ m})\left(-1.00\frac{\text{rad}}{\text{s}}\right)^2$$

$$\boldsymbol{a_c}=\mathbf{0.375\frac{m}{s^2}}$$

7d) Use

$$a_{total}=\sqrt{a_c^2+a_{tan}^2}$$

$$a_{total}=\sqrt{a_c^2+(r\alpha)^2}$$

$$a_{total}=\sqrt{\left(0.375\frac{\text{m}}{\text{s}^2}\right)^2+\left((0.375\text{ m})\left(-2.00\frac{\text{rad}}{\text{s}^2}\right)\right)^2}$$

$$\boldsymbol{a_{total}}=\mathbf{0.839\frac{m}{s^2}}$$

7e) I interpret this statement as asking for the angular *distance* traveled. This implies you need the area under the ωt-plot…but must take the absolute value of any negative areas!

$$angular\ distance=\left|\frac{1}{2}\left(8.0\frac{\text{rad}}{\text{s}}\right)(4.0\text{ s})\right|+\left|\frac{1}{2}\left(-2.0\frac{\text{rad}}{\text{s}}\right)(1.0\text{ s})\right|$$

$$angular\ distance=17\text{ rad}$$

$$\boldsymbol{angular\ distance}=\mathbf{2.71\ rev}$$

Note: some students correctly applied constant acceleration kinematics for this case. That said, please ensure you verify acceleration is constant before using that technique.