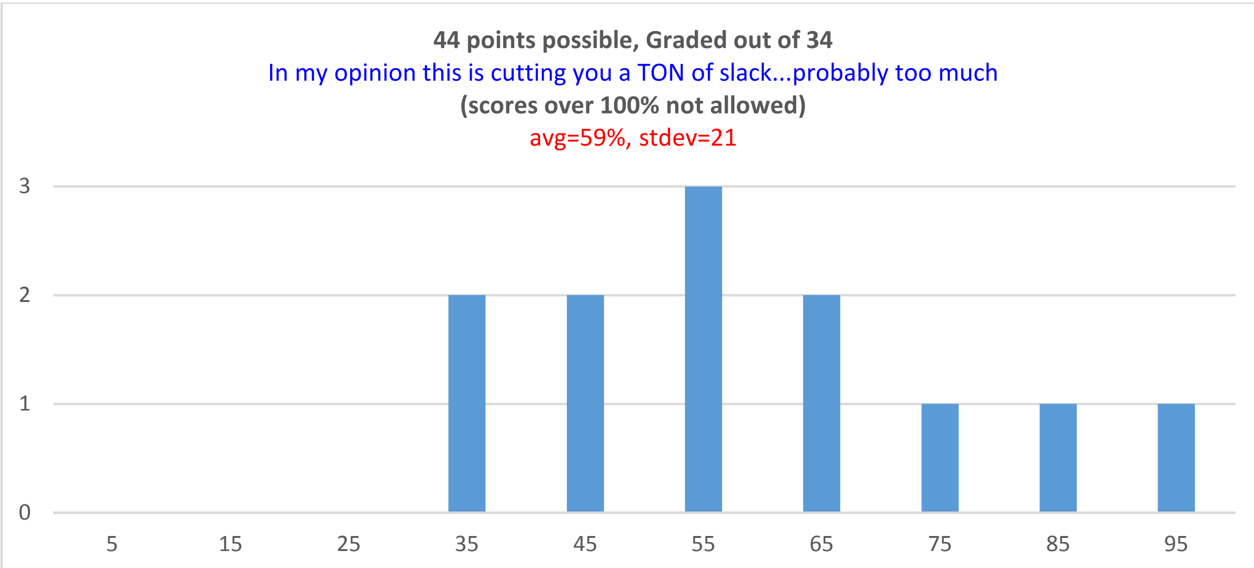
161sp22t3cSoln
Distribution on this page.
Solutions begin on the next page.



1a) Using symmetry considerations, the center of mass must be below the center of the circle.
We were told the center of the disc is the origin of a standard coordinate system.
The best answer is "On the negative y-axis".

1b) For objects with holes cut out we typically use

$$y_{cm} = \frac{y_1 m_1 - y_2 m_2}{m_1 - m_2}$$

When using this equation in this particular scenario:

- y_1 is the center of mass position of the disc (disc WITHOUT hole in it).
- $m_1 = \sigma A_1$ where σ is area mass density and A_1 is disc area (WITHOUT hole)
- y_2 is the center of mass position of the rectangle
- $m_2 = \sigma A_2$ where σ is area mass density and A_2 is rectangle area

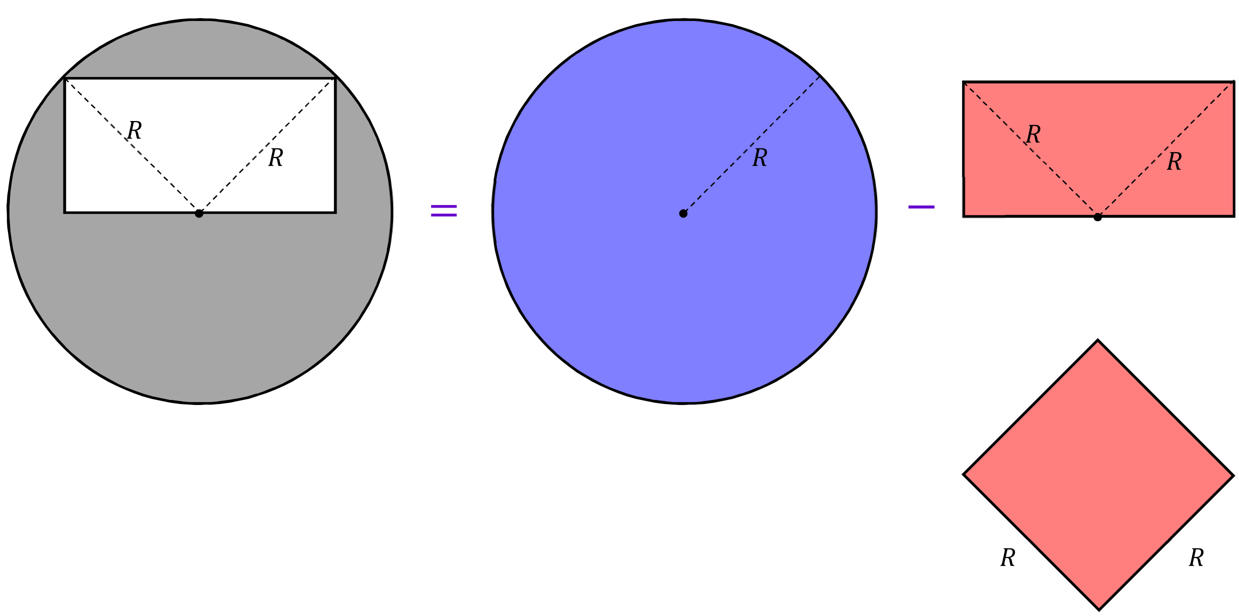
You can use a 45-45-90 triangle to determine the rectangle has width $2\left(\frac{R}{\sqrt{2}}\right) = \sqrt{2}R$ and height $\frac{R}{\sqrt{2}}$.
Notice $y_2 = \frac{1}{2}\cdot\frac{R}{\sqrt{2}}$ and $A_2 = R^2$.
A cool way to see $A_2 = R^2$ is to rearrange the rectangle into a square as shown below the final result.

$$y_{cm} = \frac{(0)(\sigma\pi R^2) - \left(\frac{1}{2}\cdot\frac{R}{\sqrt{2}}\right)\sigma\left(\sqrt{2}R\right)\left(\frac{R}{\sqrt{2}}\right)}{\sigma\pi R^2 - \sigma\left(\sqrt{2}R\right)\left(\frac{R}{\sqrt{2}}\right)}$$

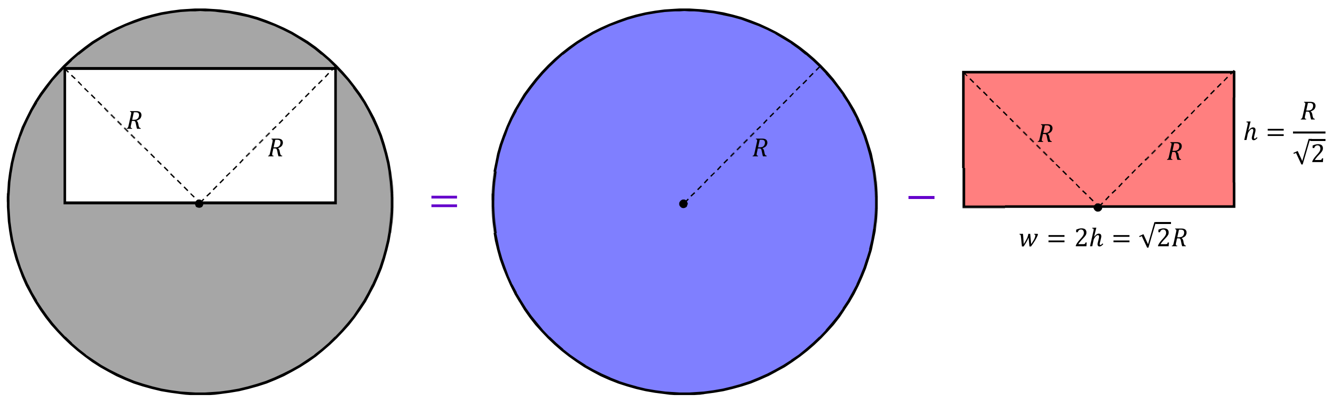
$$y_{cm} = \frac{-\sigma\left(\frac{R}{2\sqrt{2}}\right)R^2}{\sigma\pi R^2 - \sigma R^2}$$

$$y_{cm} = \frac{-\frac{1}{2\sqrt{2}}}{\pi - 1}R$$

$$\boldsymbol{y_{cm} = -0.1651R}$$



1c) I originally wanted to ask this as well but it was already a long test and something had to go.



$$I_{tot} = I_1 - I_2$$

$$I_{tot} = \left(\frac{1}{2}m_1R^2\right) - \left(I_{cm} + m_2\left(\frac{h}{2}\right)^2\right)$$

$$I_{tot} = \frac{1}{2}m_1R^2 - \left(\frac{1}{12}m_2(h^2 + w^2) + m_2\left(\frac{h}{2}\right)^2\right)$$

$$I_{tot} = \frac{1}{2}m_1R^2 - \left(\frac{5}{12}m_2h^2 + \frac{1}{4}m_2h^2\right)$$

$$I_{tot} = \frac{1}{2}m_1R^2 - \left(\frac{5}{12} + \frac{1}{4}\right)m_2h^2$$

$$I_{tot} = \frac{1}{2}(\sigma\pi R^2)R^2 - \frac{2}{3}\big(\sigma h(2h)\big)h^2$$

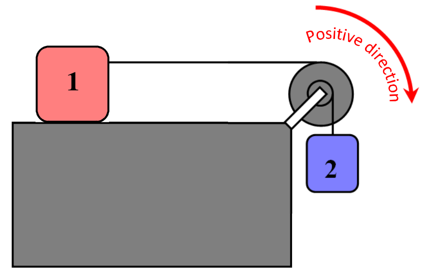
$$I_{tot} = \frac{1}{2}\sigma\pi R^4 - \frac{4}{3}\sigma h^4$$

$$I_{tot} = \frac{1}{2}\sigma\pi R^4 - \frac{4}{3}\sigma\left(\frac{R}{\sqrt{2}}\right)^4$$

$$I_{tot} = \sigma R^4\left(\frac{\pi}{2} - \frac{1}{3}\right)$$

$$\boldsymbol{I_{tot} = 1.237\sigma R^4}$$

2a) During one revolution of the pulley, the string on the outer radius must move three times as much (because it is connected at three times the radius).



We know m_1 must travel distance $3h$ if m_2 travels distance h.

2b) **First Style – Using Forces & Torque:** I will assume anything going clockwise relative to the pivot is in the positive direction. Forces on each block gives

$$T_1 = m_1 a_1$$
$$m_2 g - T_2 = m_2 a_2$$

Because the string is not slipping relative to the disk we can use $a_1 = 3R\alpha$ and $a_2 = R\alpha$.
Furthermore, to speed up the algebra I will multiply the $m_1 a_1$ equation by $3R$ and multiply the $m_2 a_2$ equation by R.

$$3RT_1 = 3Rm_1(3R\alpha)$$
$$Rm_2 g - RT_2 = Rm_2(R\alpha)$$

Using torques on the pulley gives

$$RT_2 \sin 90° - 3RT_1 \sin 90° = I\alpha$$

When adding these three equations the internal tension forces drop out!

$$Rm_2 g = (I + 9m_1 R^2 + m_2 R^2)\alpha$$

The moment of inertia is

$$I = \frac{1}{2} m_p r_{disk}^2 = \frac{1}{2}(4m)(3R)^2 = 18mR^2$$

Recall the masses were $m_1 = m$ & $m_2 = 2m$.

$$R(2m)g = (18mR^2 + 9mR^2 + (2m)R^2)\alpha$$

From here we find

$$\alpha = \frac{2g}{29R}$$

To get *translational* acceleration of m_2 we must multiply by the radius R (because m_2 connects at radius R).

$$a_2 = \frac{2}{29} g \approx 0.676 \frac{\text{m}}{\text{s}^2}$$

Alternate style using energy and kinematics:
Watch out! The speed of m_1 is three times the speed of m_2 (same reasoning as in part 2a)!

$$m_2 gh = \frac{1}{2} m_1 (3v)^2 + \frac{1}{2} m_2 v^2 + \frac{1}{2} I\omega^2$$

Finally, we know rotation rate ω is linked to the speed of m_2 using radius R (we know $v = R\omega$).
Plugging it all in gives

$$2mgh = \frac{1}{2} m(3v)^2 + \frac{1}{2}(2m)v^2 + \frac{1}{2}(18mR^2)\omega^2$$

$$4mgh = 9mv^2 + 2mv^2 + 18mv^2$$

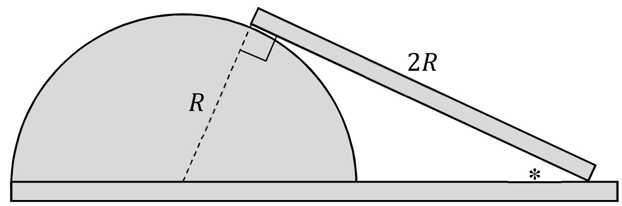
$$4gh = 29v^2$$

$$v = \sqrt{\frac{4}{29} gh}$$

From there use $v_f^2 = v_i^2 + 2ah$ to get the same result as before.

2c) If both masses were connected at the smaller radius, m_1 would move slower (no longer moving three times as fast as m_2). With less energy required for m_1, we expect a larger final rotation rate!

3a) $* = \tan^{-1}\left(\frac{R}{2R}\right) = 26.\underline{5}7°$



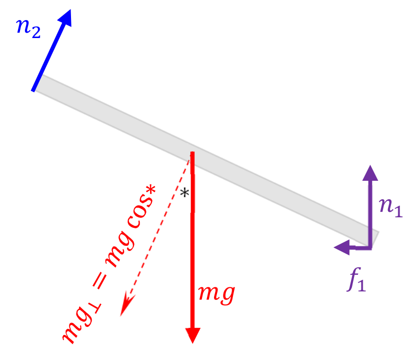
3b) Torques about the point of contact with the floor implies the magnitude of torque from the n_2 must balance the magnitude of torque from the rod's weight. Using the *upper* FBD in the figure at right I found

$$Ln_2 = \frac{L}{2}mg\cos*$$

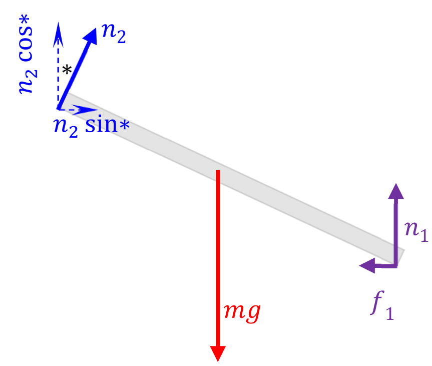
From there I used the angle in part 1 to find

$$\boldsymbol{n_2 = 0.44\underline{7}2mg}$$

I will keep the extra digit since I know I need to use this number in subsequent calculations.



1c) Now I split up all forces according to a standard coordinate system (*lower* FBD shown at right).



Forces in the horizontal direction give

$$f_1 = n_2\sin*$$

Forces in the vertical direction give

$$n_1 + n_2\cos* = mg$$

Determination of the *minimum* coefficient of friction implies we should assume the rod is on the verge of slipping ($f_1 = \mu_s n_1$).

$$\mu_s n_1 = n_2\sin*$$

$$\mu_s(mg - n_2\cos*) = n_2\sin*$$

$$\mu_s = \frac{n_2\sin*}{mg - n_2\cos*}$$

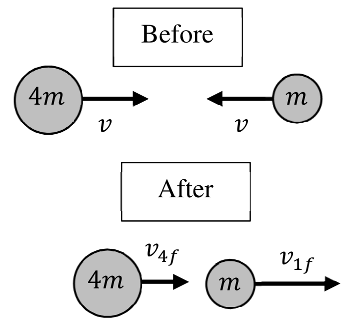
$$\mu_s = \frac{\sin*}{\frac{mg}{n_2} - \cos*}$$

$$\mu_s = \frac{\sin(26.\underline{5}7°)}{\frac{mg}{0.44\underline{7}2mg} - \cos(26.\underline{5}7°)}$$

$$\boldsymbol{\mu_s = 0.333}$$

Tip: rather than getting the angle, it may reduce rounding errors to use $\sin* = \frac{R}{\sqrt{5}R} = \frac{1}{\sqrt{5}}$ and $\cos* = \frac{2R}{\sqrt{5}R} = \frac{2}{\sqrt{5}}$.

4a) We were told this is was an *elastic* collision.
This implies both momentum and energy are conserved.



Note, I have no clue which way the masses will move after the collision.
What do I do in this instance?

- Assume they both move to the right after the collision.
- If I get a negative value for a v_f, I know that mass actually moves to the *left*.
- I get the *speed* to the *left* by taking the absolute value of my result.

Momentum Conservation	Energy Conservation (ignore the ½'s since they all cancel here)
$(4m)v - (m)v = (4m)v_{4f} + (m)v_{1f}$ Since we are finding v_{4f} I will choose to isolate v_{1f} (so I eliminate v_{1f} when I plug into the energy eqt'n). $v_{1f} = 3v - 4v_{4f}$	$(4m)v^2 + (m)v^2 = (4m)v_{4f}^2 + (m)v_{1f}^2$ Think: there should never be any minus signs here since kinetic energy is always positive! $5v^2 = 4v_{4f}^2 + v_{1f}^2$

Plug in v_{1f} from momentum conservation into the energy equation…

$$5v^2 = 4v_{4f}^2 + \left(3v - 4v_{4f}\right)^2$$

$$5v^2 = 4v_{4f}^2 + 9v^2 + 16v_{4f}^2 - 24vv_{4f}$$

$$0 = 20v_{4f}^2 - 24vv_{4f} + 4v^2$$

Divide all terms by 4 then try to factor.
Tip: when factoring, one root will always be $v_{4f} - v_{4i}$. In this case that root would be $v_{4f} - v$.
Why does this root always appear?
If no collision happens, energy would be conserved. This is that uninteresting (but mathematically valid) situation.

$$0 = 5v_{4f}^2 - 6vv_{4f} + v^2$$

$$0 = (v_{4f} - v)(5v_{4f} - v)$$

At this point we know either $v_{4f} = v$ or $v_{4f} = \frac{v}{5}$.
As I stated before, the first root would describe a situation where no collision occurred.
There fore the second root must be the correct answer.

$$\boldsymbol{v_{4f} = \frac{v}{5}}$$

4b) During the collision, both objects experience the same change in momentum (magnitude).

4c) During the collision, both objects experience the same force magnitude (Newton's third law).

4d) During the collision, the lighter mass m has more acceleration magnitude (same force, less mass, more a).

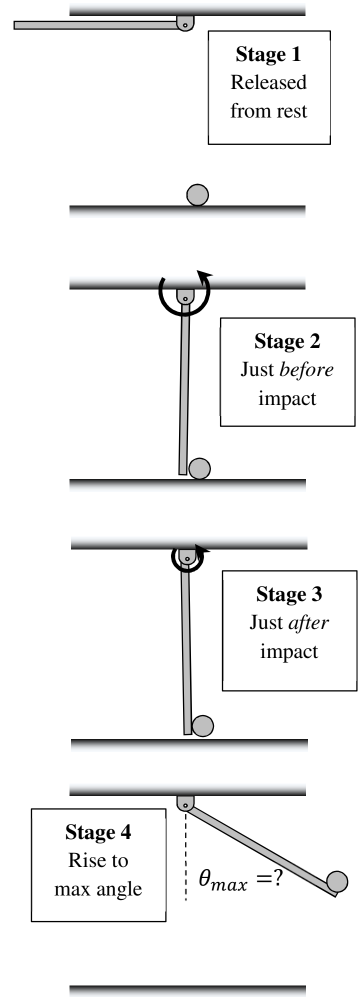
5a) Use energy methods to determine rotation rate just *before* impact ($E_1 = E_2$).

$$m_{rod} g h_{\substack{rod \\ cm \\ 1}} = \frac{1}{2} I \omega_2^2$$

$$m_{rod} g \left(\frac{d}{2}\right) = \frac{1}{2}\left(\frac{1}{3} m_{rod} d^2\right) \omega_2^2$$

$$gd = \left(\frac{1}{3} d^2\right) \omega^2$$

$$\boldsymbol{\omega_2 = \sqrt{\frac{3g}{d}}}$$



5b) Use

$$r_{cm} = \frac{r_{\substack{rod \\ cm}} m_{rod} + r_{pm} m_{pm}}{m_{rod} + m_{pm}} = \frac{\left(\frac{d}{2}\right)(3m) + (d)(m)}{4m} = \boldsymbol{\frac{5}{8} d = 0.625d}$$

Think: this makes sense…it should be slightly lower than the halfway point.

5c) Moment of inertia just after collision is given by

$$I = I_{rod} + I_{pm} = \frac{1}{3} m_{rod} d^2 + m_{pm} d^2 = 2md^2$$

5d) Use angular momentum to compare just *before* impact to just *after* impact ($\vec{L}_2 = \vec{L}_3$).

$$I_2 \omega_2 = I_3 \omega_3$$

$$\left(\frac{1}{3}(3m) d^2\right) \sqrt{\frac{3g}{d}} = (2md^2) \omega_3$$

$$\sqrt{\frac{3g}{d}} = 2\omega_3$$

$$\omega_3 = \sqrt{\frac{3g}{4d}}$$

Now use energy as the combined object rises to some max angle ($E_3 = E_4$).

$$\frac{1}{2} I_3 \omega_3^2 = m_{tot} g r_{\substack{cm \\ 4}} (1 - \cos\theta_{max})$$

$$\frac{1}{2}(2md^2)\left(\frac{3g}{4d}\right) = (4m) g (0.625d)(1 - \cos\theta_{max})$$

$$\left(\frac{3}{4}\right) = (4)(0.625)(1 - \cos\theta_{max})$$

$$\boldsymbol{\theta_{max} = \cos^{-1}\left(1 - \frac{3}{16(0.625)}\right) = 45.6°}$$

6a) Watch out for 2π in the basement! Be sure you punch it in correctly. The conversion is

$$20.0\frac{\text{rad}}{\text{s}}\times\frac{1\text{ rev}}{2\pi\text{ rad}}\times\frac{60\text{ s}}{1\text{ min}}=\mathbf{191.0\ RPM}$$

6b) WATCH OUT! Angular acceleration is NOT constant; do NOT use constant acceleration kinematics here.
WATCH OUT! When integrating, don't forget about the initial angular velocity!
In this case, because α is a function of *time*, there is no need to separate before you integrate.
Finally, there is no need to carry the units along during the calculation as long as you pay attention to those things when writing the final answer…

$$\Delta\omega=\int_i^f -30.0t^2\ dt$$

$$\omega_f-\omega_i=[-10.00t^3]_i^f$$

$$\omega_f=\omega_i-10.00t_f^3+10.00t_i^3$$

By setting $t_i=0$ we can then set $t_f=t$ and associate ω_f with $\omega(t)$…the rotation rate as a function of time.

$$\boldsymbol{\omega(t)=\omega_f=20.0-10.00t^3}$$

For this problem I said it was ok to leave off units to reduce clutter in the answer box.
I am also ok with leaving off the $\hat{j}$ or $\hat{k}$ here since we restrict ourselves to a single axis of rotation in our class.
In general, always be prepared to include units on numerical constants in these types of formulas (ones with both numerical constants and algebraic parameters).

6c) I will use an absolute value on this result since the question asked for the *magnitude* of a_{tan}…

$$a_{tan}=|r\alpha|$$

Here $r=0.360$ m since the point of interest is on the rim of the spherical shell (at the equator so to speak).
This time I will include the units so I can check my units as a I work…

$$a_{tan}=\left|(0.360\text{ m})\left(\left(-30.0\frac{\text{rad}}{\text{s}^4}\right)(2.00\text{ s})^2\right)\right|$$

$$\boldsymbol{a_{tan}=43.2\frac{m}{s^2}}$$

Notice the units of radians in the numerator are absorbed when multiplying by the units of meters…

6d) Centripetal acceleration *magnitude* is given by

$$a_c=r\omega^2$$

Get $\omega=20.0-10.00t^3=-60.0\frac{\text{rad}}{\text{s}}$ from the function $\omega(t)$ at $t=2.00$ s.

$$a_c=(0.360\text{ m})\left(-60.0\frac{\text{rad}}{\text{s}}\right)^2$$

$$\boldsymbol{a_c=1296\frac{m}{s^2}}$$

Again, the units of radians in the numerator are absorbed when multiplying by the units of meters…

Solution continues on the next page…

6e) We know $\omega(t) = 0$ at the instant the shell reverses direction.
Set $\omega(t) = 0$ and solve for t.
After wards use that time in an equation for $\theta(t)$.
WATCH OUT! Because angular acceleration is a function of time…it is NOT constant.
One cannot use constant angular acceleration kinematics.
Integrate using $\Delta\theta = \int_i^f \omega(t)\ dt$.
Finally, don't forget to convert the final result for θ to the correct units of revolutions.

I found $t = \sqrt[3]{2} = 1.2599$ s.
I keep an extra digit to avoid intermediate rounding.
Think: we usually keep four digits on a three digit number to avoid intermediate rounding.
Since we typically keep an extra digit when the 1st digit is a 1, this implies keep 5 digits instead of rounding to 4.

To get $\theta(t)$

$$\Delta\theta = \int_i^f \omega(t)\ dt$$

$$\theta(t) = \theta_i + \int_i^f (20.0 - 10.00t^3)\ dt$$

We were told

$$\theta(t) = 20.0t - 2.5t^4$$

I used this & $t = 1.2599$ s to determine

$$\theta = 18.899 \text{ rad}$$

Finally, I convert

$$\theta = 18.899 \text{ rad} \times \frac{1 \text{ rev}}{2\pi \text{ rad}} = \mathbf{3.01\ rev}$$