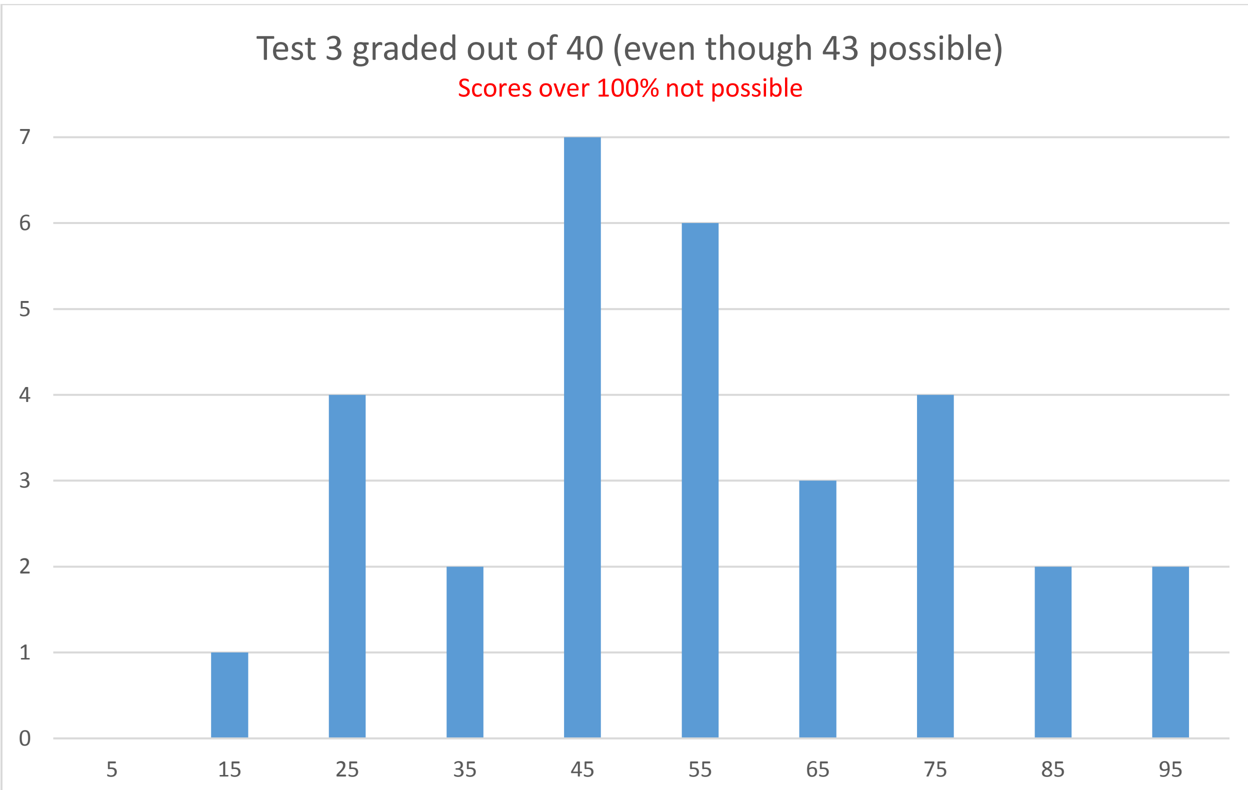
**161sp23t2asoln**

Distribution on this page.  Solution begins on the next page.

Note: one very unusual aspect of this exam is that I ended up having to cut the angular momentum problem due to time constraints. In general I try to have a problem from every chapter.  Gee…I wonder if an angular momentum problem is on the Spring 2023 Final???



1a) & 1b) & 1c) I pretty much ask some variation of this every year.
I would actually do part 1c first, then 1a, and finally 1b.

1c) When two objects collide **the forces they exert on each other have equal magnitude** in opposite directions.

$$\vec{F}_{1on2} = -\vec{F}_{2on1}$$

1a) Typically other forces are negligible during collisions. In essence, we typically assume the force exerted by one object on the other is approximately equal to the net force on the other object. Newton's 2nd law for object 2 gives

$$\vec{F}_{NETon2} = m_2\vec{a}_2$$
$$\vec{F}_{1on2} \approx m_2\vec{a}_2$$
$$\vec{a}_2 \approx \frac{\vec{F}_{1on2}}{m_2}$$

Similarly, Newton's 2nd law for object 1 gives

$$\vec{a}_1 \approx \frac{\vec{F}_{2on1}}{m_1}$$

**If net force magnitude is the same on each object, the smaller mass has more acceleration (magnitude)**.

1b) Finally, recall that momentum relates to force using

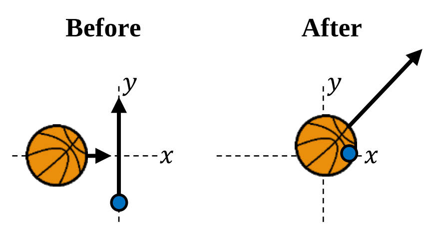
$$\vec{F}_{NETon2} = \frac{d}{dt}\vec{p}_2 \approx \frac{\Delta\vec{p}_2}{\Delta t}$$
$$\vec{F}_{1on2} \approx \frac{\Delta\vec{p}_2}{\Delta t}$$
$$\Delta\vec{p}_2 \approx \vec{F}_{1on2}\,\Delta t$$

Here $\Delta t$ is the collision time interval. Similarly

$$\Delta\vec{p}_1 \approx \vec{F}_{2on1}\,\Delta t$$

If the forces have equal magnitude **we expect equal magnitude change in momentum** as well!

1d) We are told the collision is perfectly inelastic.
This implies the objects stick together and move in unison after the collision.




Momentum in the $x$-direction is conserved.
Initially the basketball-sphere system has positive $x$-direction momentum.
The system continues to have positive $x$-direction momentum after the collision.

Momentum in the $y$-direction is also conserved.
Initially the basketball-sphere system has positive $y$-direction momentum.
The system continues to have positive $y$-direction momentum after the collision.

**The combined basketball-sphere object moves somewhere into the 1st quadrant**.

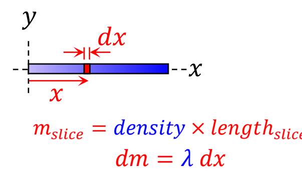
2a) Linear mass density increases with horizontal position according to the density equation

$$\lambda = cx^2$$

We expect the right end of the rod to have more mass than the left end.

**The center of mass position should be closer to the right end of the rod.**

2b) The moment of inertia is found using




$$I_{yy} = I_{about\ y-axis} = \int \left(radius_{from\ y-axis}\right)^2 dm$$

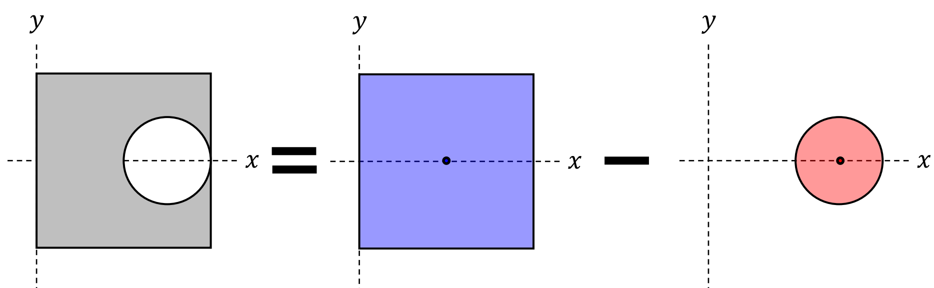
$$I = \int x^2\, (\lambda\, dx)$$

$$I = \int_0^L x^2\, (cx^2\, dx)$$

$$I = \frac{cL^5}{5}$$

$$\boldsymbol{c = \frac{5I}{L^5}}$$

3a) & 3b) The system can be visualized as a solid square plate minus a solid circular plate.




$$M_{total} = m_1 - m_2 \qquad x_{CM} = \frac{x_1 m_1 - x_2 m_2}{m_1 - m_2}$$

In these equations

- $x_1 = \frac{s}{2}$ is the horizontal coordinate of the center of mass of object 1 (square plate)
- $m_1 = \sigma A_1 = \sigma s^2$ is the mass of object 1 (square plate) assuming uniform plate density $\sigma$ in units of $\frac{\mathrm{kg}}{\mathrm{m}^2}$
- $x_2 = \frac{3}{4}s$ is the horizontal coordinate of the center of mass of object 2 (circular plate)
- $m_1 = \sigma A_2 = \sigma \frac{\pi s^2}{16}$ is the mass of object 2 (circular plate) assuming uniform plate density $\sigma$ in units of $\frac{\mathrm{kg}}{\mathrm{m}^2}$
- If you care, the mass per unit area ($\sigma$) relates to standard 3D density ($\rho$) using

$$\sigma = \rho \times (plate\ thickness)$$

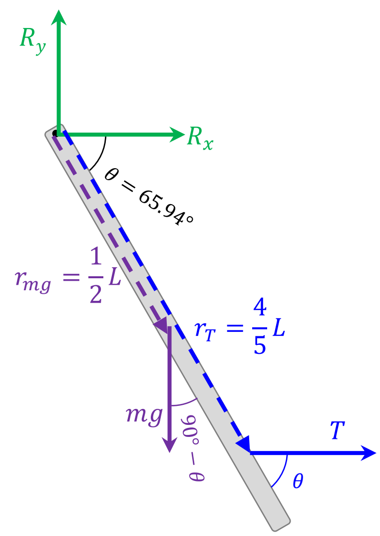
Total mass of the plate is

$$M_{total} = m_1 - m_2 = \sigma s^2 - \sigma \frac{\pi s^2}{16} = 0.80\underline{3}7\sigma s^2$$

Notice every term in the center of mass equation has $\sigma$…mass density will drop out in that result.

$$x_{CM} = \frac{x_1 m_1 - x_2 m_2}{m_1 - m_2} = \frac{\left(\frac{s}{2}\right)(\sigma s^2) - \left(\frac{3}{4}s\right)\left(\sigma \frac{\pi s^2}{16}\right)}{\sigma s^2 - \sigma \frac{\pi s^2}{16}} = 0.439s$$

4a) & 4b) I chose to draw my FBD like the one shown at right.
Currently my style is to think as little as possible about the reaction forces at the pivot.
I blindly choose them to be $R_x$ to the right and $R_y$ upwards.
If I get a *negative result*, the true direction is *opposite the direction drawn*.




Sum of torques about the pivot (using CCW as positive direction) gives

$$r_T T \sin\theta_T - r_{mg} mg \sin\theta_{mg} = 0$$

Recall, this is a static equilibrium problem… $\alpha = 0$.

$$\left(\frac{4}{5}L\right) T \sin\theta - \frac{L}{2} mg \sin(90° - \theta) = 0$$

Solving for $m$:

$$m = \frac{8T}{5g}\tan\theta$$

$$\boldsymbol{m \approx 3.5\underline{8}3\frac{T}{g}}$$

Sum of forces gives

| Horizontal Force Equation | Vertical Force Equation |
|---|---|
| $R_x = -T$ | $R_y = mg$<br>$R_y \approx \left(3.5\underline{8}3\frac{T}{g}\right) g$<br>$R_y \approx 3.5\underline{8}3T$ |

Putting the results together in Cartesian form gives

$$\vec{R} = \left(-1.00\hat{\imath} + 3.5\underline{8}3\hat{\jmath}\right)T$$

The *magnitude* of this force vector is

$$\boldsymbol{R \approx 3.72T}$$

5a) WATCH OUT when converting to RPM.
Many times students forget to put parentheses around $2\pi$ in the denominator.

$$0.925\,\frac{\text{rad}}{\text{s}} \times \frac{1\text{ rev}}{2\pi\text{ rad}} \times \frac{60\text{ s}}{1\text{ min}} \approx \mathbf{8.83\ RPM}$$

As a useful check, I keep in mind this conversion should change the number by *approximately* a factor of 10…
Strictly speaking the initial rotation VECTOR is shown as $\vec{\omega}_i = \mathbf{8.83\ RPM}(-\hat{\jmath})$.

Assuming you did the rest of the problem correctly I was fine with it if you flipped *all* negative signs…

5b) Several good ways to go about this. Since we have constant angular acceleration:

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

Here $\Delta\vec{\theta} = 0.100\text{ rev}(-\hat{\jmath}) = 0.62\underline{8}3\text{ rad}(-\hat{\jmath})$ while $\omega_f = 0$ since the disk comes to rest.

$$0 = \omega_i^2 + 2\alpha\Delta\theta$$

$$\alpha = -\frac{\omega_i^2}{2\Delta\theta}$$

$$\boldsymbol{\alpha \approx \pm 0.68\underline{0}9\ \frac{rad}{s^2}}$$

If you assumed the original rotation direction was positive you should have the minus sign on this result.

5c) Because angular acceleration is constant we can use

$$\omega_f = \omega_i + \alpha t$$

$$t = \frac{\omega_f - \omega_i}{\alpha}$$

$$\boldsymbol{t \approx 1.359\ s}$$

5d) INITIAL total acceleration (magnitude) is given by

$$a_{total\ init} = \sqrt{a_{tan\ init}^2 + a_{c\ init}^2}$$

$$a_{total\ init} = \sqrt{(r\alpha)^2 + (r\omega_i^2)^2}$$

$$a_{total\ init} = r\sqrt{\alpha^2 + \omega_i^4}$$

In this equation, $r$ means distance from axis to the point of interest where one wishes to determine $a_{total}$.
Since our point of interest is on the edge of the disk, this distance is the full radius of the disk (0.375 m).
Finally, notice one must use units of radians for angular quantities $\alpha$ & $\omega_i$ for the units to work out properly!

$$a_{total\ init} \approx (0.375\text{ m})\sqrt{\left(-0.68\underline{0}9\ \frac{\text{rad}}{\text{s}^2}\right)^2 + \left(0.925\,\frac{\text{rad}}{\text{s}}\right)^4}$$

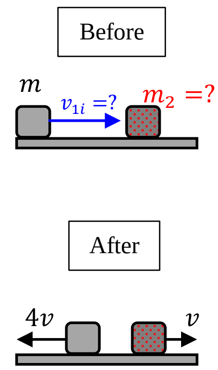
$$\boldsymbol{a_{total\ init} \approx 0.410\,\frac{m}{s^2}}$$

5e) **WATCH OUT!**
Notice the disk is being rotated about an axis in the plane of the disk.
Notice $I = \frac{1}{4}mr^2$ instead of the usual $I = \frac{1}{2}mr^2$ used so often in rolling motion problems!

$$RKE_i = \frac{1}{2}I\omega_i^2 = \frac{1}{2}\left(\frac{1}{4}mr^2\right)\omega_i^2 \approx \frac{1}{8}(2.00\text{ kg})(0.375\text{ m})^2\left(0.925\,\frac{\text{rad}}{\text{s}}\right)^2 = \mathbf{30.1\ mJ}$$

Again, notice one must use units of $\frac{\text{rad}}{\text{s}}$ on $\omega_i$ for the units to work out properly.

6a) & 6b) I chose to label my figure as shown at right.
In my figure, I am assuming $v's$ are speeds with velocity directions indicated by arrows.
Other reasonable styles are possible and should give the same results as mine shown below.




For *every* type of collision we typically use conservation of momentum. We can do this because assume collision time is short enough such that external forces are negligible (usually a good assumption in the real world).

For elastic collisions we can also assume energy is conserved.

| **Conservation of Momentum** | **Conservation of Energy** |
|---|---|
| $mv_{1i} = m(-4v) + m_2 v$ | $\frac{1}{2}mv_{1i}^2 = \frac{1}{2}m(4v)^2 + \frac{1}{2}m_2 v^2$ |

For me it seemed easiest to divide all terms in both equations by $m$.
I also cancelled the $\frac{1}{2}'s$ in the energy equation right away.

| **Conservation of Momentum** | **Conservation of Energy** |
|---|---|
| $v_{1i} = -4v + \frac{m_2}{m}v$ <br> $v_{1i} = v\left(\frac{m_2}{m} - 4\right)$ | $v_{1i}^2 = (4v)^2 + \frac{m_2}{m}v^2$ |

Now I sub in the equation for $v_{1i}$ from conservation of momentum into the energy equation and solve.

$$v^2\left(\frac{m_2}{m} - 4\right)^2 = (4v)^2 + \frac{m_2}{m}v^2$$

Notice $v^2$ cancels in every term!!!

$$\left(\frac{m_2}{m} - 4\right)^2 = 16 + \frac{m_2}{m}$$

$$\left(\frac{m_2}{m}\right)^2 + \left(-8\frac{m_2}{m}\right) + 16 = 16 + \frac{m_2}{m}$$

$$\left(\frac{m_2}{m}\right)^2 + \left(-9\frac{m_2}{m}\right) = 0$$

$$\boldsymbol{m_2 = 9.00m}$$

Now plug this result back into

$$v_{1i} = v\left(\frac{m_2}{m} - 4\right)$$

$$v_{1i} = v\left(\frac{\mathbf{9m}}{m} - 4\right)$$

$$\boldsymbol{v_{1i} = 5.00v}$$

7a) **TRUE.** Acceleration is constant when rolling without slipping in a straight line.
Note: acceleration is not constant if the ramp was a circular arc or if we had a pendulum swinging…

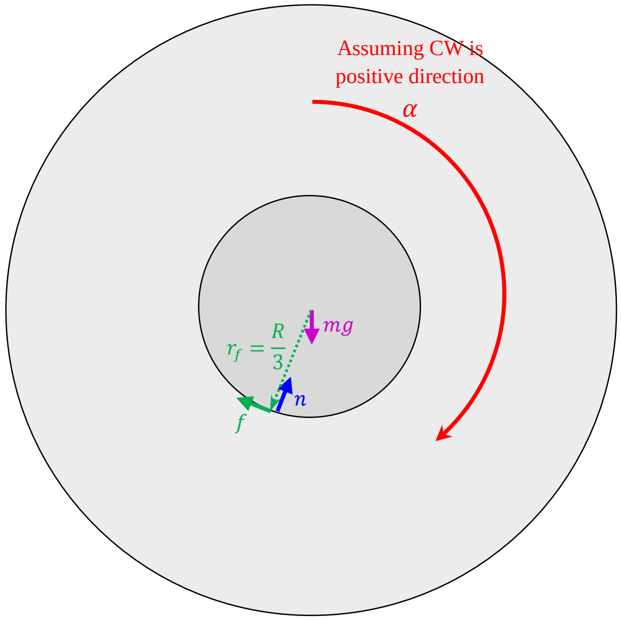
7b) $\boldsymbol{v = \frac{R\omega}{3}}$
The radius which relates translational and rotational speeds is the radius connected to the ramp/road/string.
We may use this result for rolling *without* slipping.

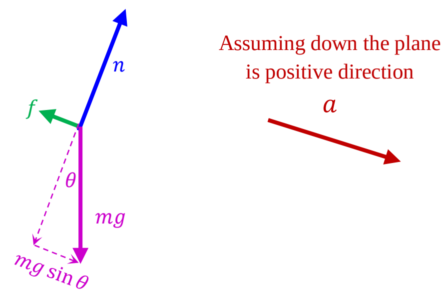
I put part 7c on the next page so I could draw a huge diagram for you to see things better…

7c) Rolling motion implies both translation and rotation occur. Do both forces and torques.

I'll choose to do torques about the center of mass (since that technique also works for rolling *with* slipping).

On next page I will show the instantaneous pivot method…




<table>
<tr><th>Sum of torques about center of mass</th><th>Sum of forces down the plane</th></tr>
<tr><td>

Normal force and weight cause no torque since they have lines of action through the pivot.

$$\Sigma\tau_{CM} = I_{CM}\alpha$$
$$\tau_f = I_{CM}\alpha$$
$$r_f f \sin\theta_f = I_{CM}\alpha$$
$$\frac{R}{3} f \sin 90° = I_{CM}\alpha$$
$$\frac{Rf}{3} = I_{CM}\alpha$$

This disk rotates about an axis *perpendicular to the plane*:

$$I_{cm} = \frac{1}{2}mR^2$$

Note: in this problem we were told the extra disks produce negligible contribution to the moment of inertia.

$$\frac{R}{3} f = \frac{1}{2}mR^2\alpha$$

</td><td>

$$mg\sin\theta - f = ma$$
$$f = mg\sin\theta - ma$$

Since rolling *without* slipping can use $a = r\alpha$.

Use $r = \frac{R}{3}$ since that is the radius touching the ramp!

$$f = mg\sin\theta - m\frac{R}{3}\alpha$$

Now plug this result for $f$ into torque equation…

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$$\frac{R}{3}\left(mg\sin\theta - m\frac{R}{3}\alpha\right) = \frac{1}{2}mR^2\alpha$$

Group like terms to solve for $\alpha$. A common mistake is to forget to use $a = r\alpha$. If you forget this, the final result contains a circular reference and is not considered in good form.

$$\frac{R}{3}mg\sin\theta - m\frac{R^2}{9}\alpha = \frac{1}{2}mR^2\alpha$$
$$\frac{R}{3}mg\sin\theta = \frac{11}{18}mR^2\alpha$$
$$\boldsymbol{\alpha = \frac{6g\sin\theta}{11R} \approx \frac{0.545g\sin\theta}{R}}$$

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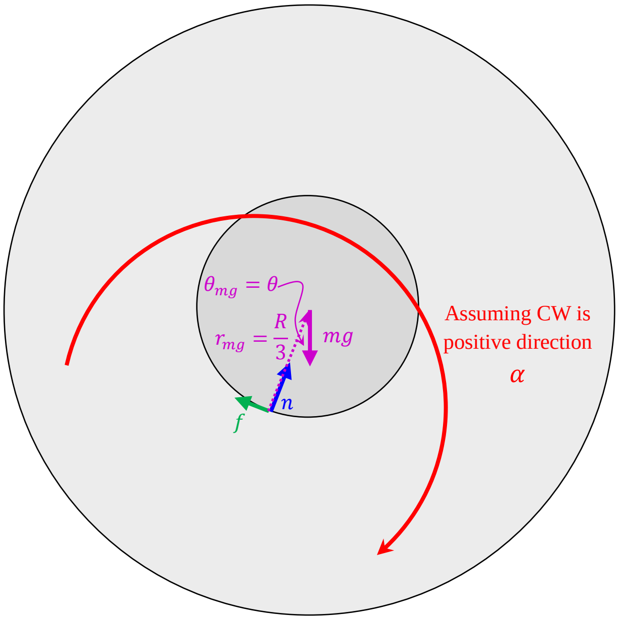



**Sum of torques about point of contact (instantaneous pivot)**

Normal force and friction cause no torque since they have lines of action through the pivot.

$$\Sigma\tau_{CM} = I_{inst}\alpha$$

$$\tau_{mg} = (I_{CM} + md^2)\alpha$$

$$r_{mg} mg \sin\theta_{mg} = \left(I_{CM} + m\left(\frac{R}{3}\right)^2\right)\alpha$$

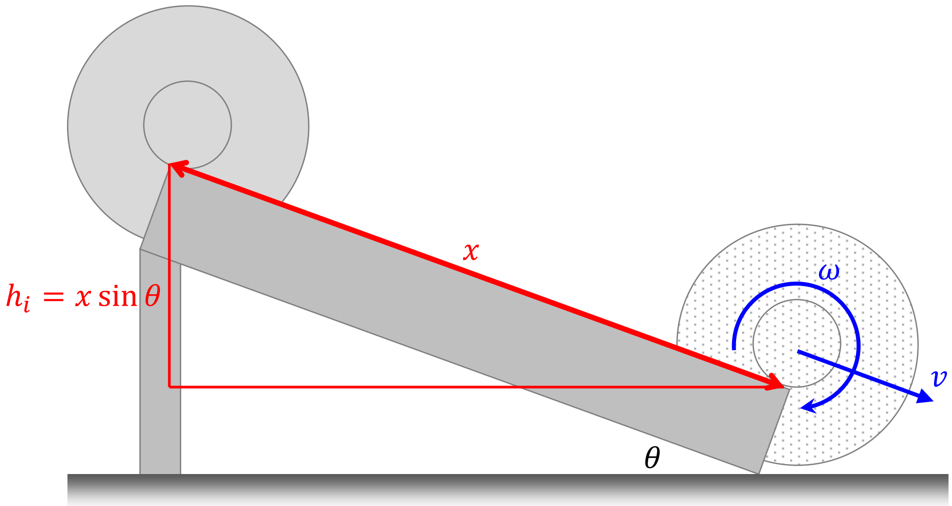
$$\frac{R}{3} mg \sin\theta = \left(\frac{1}{2}mR^2 + m\left(\frac{R}{3}\right)^2\right)\alpha$$

$$\frac{R}{3} mg \sin\theta = \left(\frac{1}{2} + \frac{1}{9}\right) mR^2\alpha$$

$$\frac{R}{3} mg \sin\theta = \frac{11}{18} mR^2\alpha$$

$$\boldsymbol{\alpha = \frac{6g\sin\theta}{11R} \approx \frac{0.545g\sin\theta}{R}}$$

7d) At this point, you could actually use constant acceleration kinematics since acceleration is actually constant. I prefer to show you energy methods since that technique works even when acceleration non-constant. I will then check the result using kinematics.




In this solution I will use $RKE$ for rotational kinetic energy and $TKE$ for translational kinetic energy.

$$RKE_i + TKE_i + U_{grav\ i} + W_{\substack{ext\\non-con}} = RKE_f + TKE_f + U_{grav\ f}$$

When rolling without slipping, friction does no work! Recall, the point of contact with the road does NOT slip relative to the road.This frictional force is present but there is no displacement relative to the road.
Normal force is perpendicular to displacement, no work from that force either.
Starting from rest. Letting lowest point (final position) in the problem have zero height.

$$0 + 0 + mgx\sin\theta + 0 = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 + 0$$

$$2mgx\sin\theta = I\omega^2 + mv^2$$

Divide all by $m$ since it tends to simplify algebra later on…

$$2gx\sin\theta = \frac{I}{m}\omega^2 + v^2$$

Remember, rewrite $\omega$ using $v = \frac{R}{3}\omega \rightarrow \quad \omega = \frac{3v}{R}$.
It is bad form to have our final answer for $v$ in terms of omega (this is a circular reference).

$$2gx\sin\theta = \frac{I}{m}\left(\frac{3v}{R}\right)^2 + v^2$$

$$2gx\sin\theta = v^2\left(\frac{9I}{mR^2} + 1\right)$$

$$2gx\sin\theta = v^2\left(\frac{9\left(\frac{1}{2}mR^2\right)}{mR^2} + 1\right)$$

$$2gx\sin\theta = v^2\left(\frac{11}{2}\right)$$

$$\boldsymbol{v = \sqrt{\frac{4}{11}gx\sin\theta} \approx 0.603\sqrt{gx\sin\theta}}$$

Kinematics check on the next page…

7d) Continued. Check previous result using kinematics:

$$v_f^2 = v_i^2 + 2a\Delta x$$

$$v^2 = 0 + 2\left(\frac{R}{3}\alpha\right)x$$

$$v = \sqrt{\frac{2Rx\alpha}{3}}$$

Plug in the result for $\alpha$ from problem 7c…

$$v = \sqrt{\frac{2Rx\left(\frac{6g\sin\theta}{11R}\right)}{3}}$$

$$v = \sqrt{\boldsymbol{\frac{4}{11}gx\sin\theta}}$$

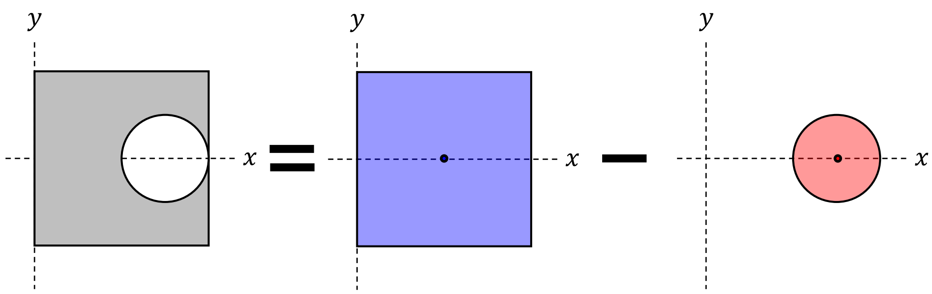
Notice we get the same result as we did using energy methods.

This is method is probably easier for this particular problem.

That said, sometimes you really want to know this technique…sometimes acceleration isn't constant.

REMEMBER THIS: you *shouldn't* use constant acceleration kinematics for a pendulum swinging…

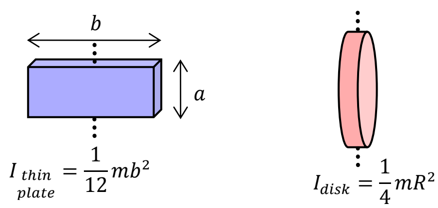
**EXTRA CREDIT:**




- $x_1 = \frac{s}{2}$ is the horizontal coordinate of the center of mass of object 1 (square plate)
- $m_1 = \sigma A_1 = \sigma s^2$ is the mass of object 1 (square plate) assuming uniform plate density $\sigma$ in units of $\frac{\text{kg}}{\text{m}^2}$
- $x_2 = \frac{3}{4}s$ is the horizontal coordinate of the center of mass of object 2 (circular plate)
- $m_1 = \sigma A_2 = \sigma \frac{\pi s^2}{16}$ is the mass of object 2 (circular plate) assuming uniform plate density $\sigma$ in units of $\frac{\text{kg}}{\text{m}^2}$
- If you care, the mass per unit area ($\sigma$) relates to standard 3D density ($\rho$) using

$$\sigma = \rho \times (plate\ thickness)$$

$$I_{yy} = I_{square} - I_{disk}$$




Remember to also account for the parallel axis theorem on both of these shapes!!!

You should use the parallel axis theorem for the thin plate using $a = b = s$ and $d = \frac{s}{2}$.

You could use the parallel axis theorem on the disk (axis in-plane) using $d = \frac{3}{4}s$.

$$I_{yy} = \left(\frac{1}{12}m_1 s^2 + m_1\left(\frac{s}{2}\right)^2\right) - \left(\frac{1}{4}m_2 r^2 + m_2 d^2\right)$$

$$I_{yy} = \left(\frac{1}{3}m_1 s^2\right) - m_2\left(\frac{1}{4}\left(\frac{s}{4}\right)^2 + \left(\frac{3}{4}s\right)^2\right)$$

$$I_{yy} = \frac{1}{3}(\sigma s^2)s^2 - \left(\sigma\frac{\pi s^2}{16}\right)\left(\frac{1}{64}s^2 + \frac{9}{16}s^2\right)$$

$$I_{yy} = 0.3333\sigma s^4 - 0.11351\sigma s^4$$

$$\boldsymbol{I_{yy} \approx 0.220\sigma s^4}$$