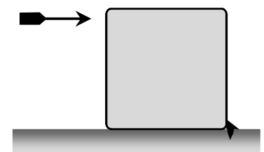
11.Nasty A bullet of unknown mass travels with speed v just before impact with a wooden plate (mass m and side length s). After impact, the bullet embeds in the top left corner of the plate. There is a small wedge affixed to the ground beside the bottom right corner of the square plate. Determine the minimum bullet mass required (with the given initial speed v) to cause the bullet to tip over and land on its side. You may assume the bottom right corner of the plate experiences negligible slipping with the ground as the plate tips over.



For ease of communication, I labeled the unknown bullet mass m_1 in the solutions.

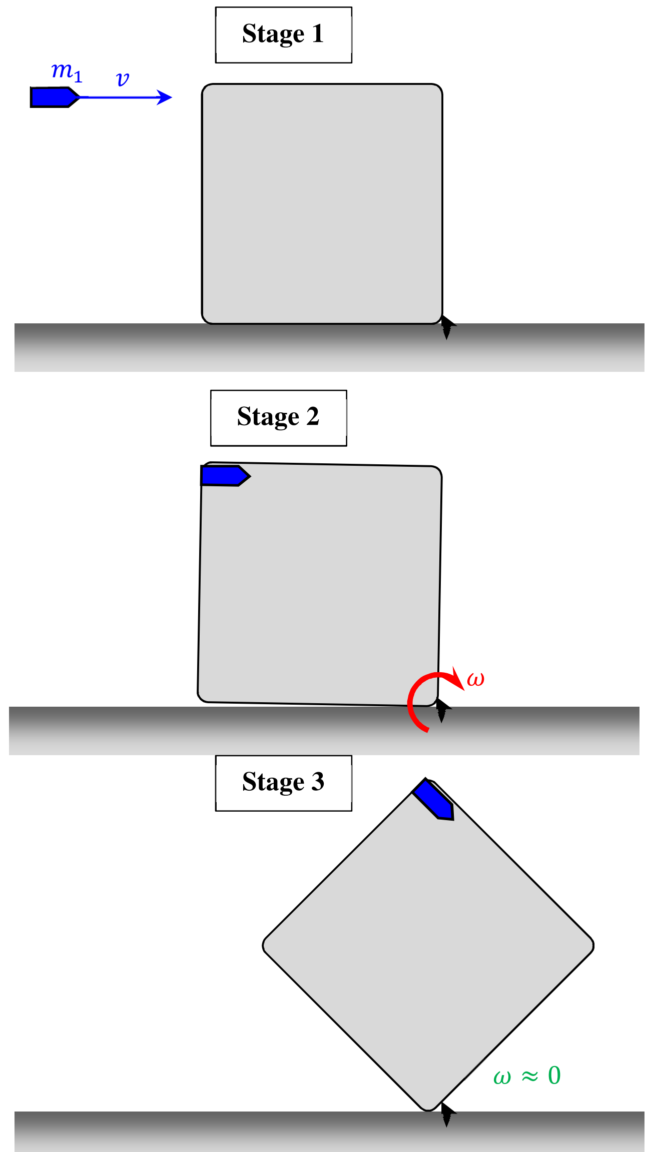
Try it without looking at the solutions.
If you get stuck, try scrolling to the next page for hint before checking the solutions.

The solution is on the pages following the hints.

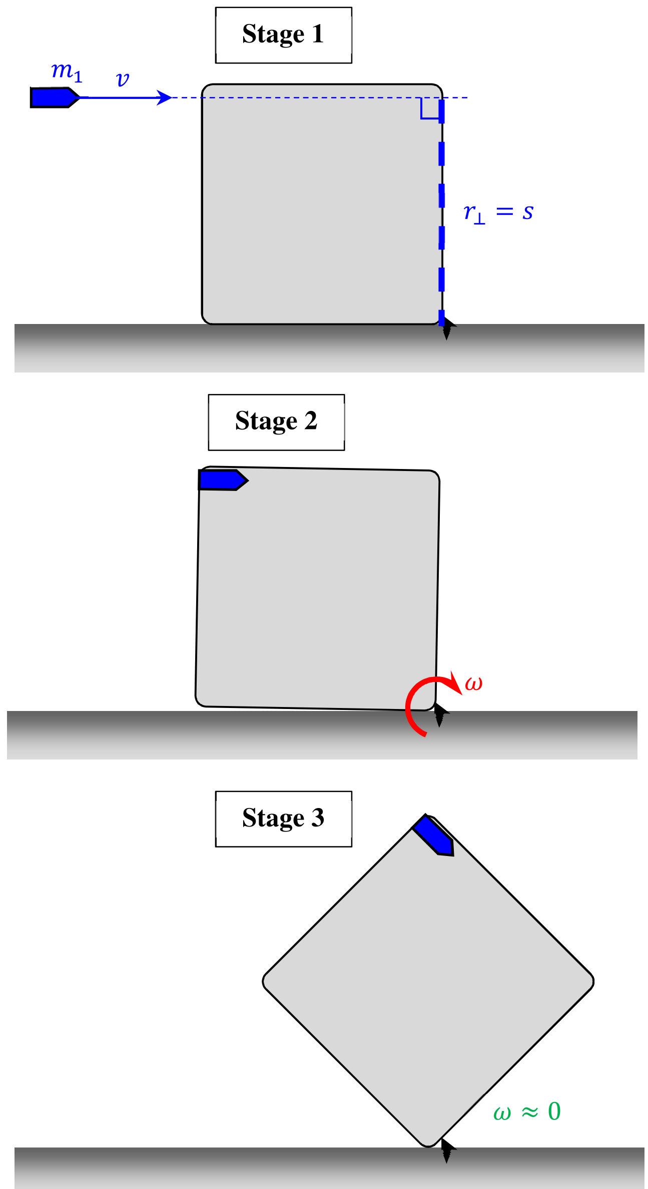
11.Nasty Hint

A lot of people recognize the need to draw several stages.

Unfortunately, many people miss the crucial stages are just *before* impact, just *after* impact, **& when the plate has rotated 45°** (not all the way over on its side). Think about it: if the plate is able to rotate 45° and is still just barely rotating, it will make it over on its side!



11.Nasty Solution



<u>When comparing stages 1 & 2</u> one should use *angular* momentum. The pivot exerts a reaction force (external force on the bullet-plate system). *Linear* momentum (a.k.a. *translational* momentum) is NOT conserved.

I will assume the pivot point is the wedge at the bottom right corner of the block.
I will assume clockwise rotation is positive.

$$\boldsymbol{L_i = L_f}$$

$$\boldsymbol{m_1 v s = I_{plate+bullet}\,\omega}$$

Compare the plate to the table of moment of inertias.
Through the center, perpendicular to the plane of the plate

$$I = \frac{1}{12}m(a^2 + b^2)$$

In this case the sides of the plate are $a = b = s$.
The pivot in our problem is shifted by half the main diagonal ($d = \frac{1}{2}(\sqrt{2}s) = \frac{s}{\sqrt{2}}$).
The bullet is essentially a point mass distance $\sqrt{2}s$ from the pivot.

$$I_{plate+bullet} = I_{CM\ plate} + m_{plate}\left(\frac{s}{\sqrt{2}}\right)^2 + m_{bullet}\left(\sqrt{2}s\right)^2$$

$$I_{plate+bullet} = \frac{1}{12}m(s^2 + s^2) + \frac{1}{2}ms^2 + 2m_1 s^2$$

$$I_{plate+bullet} = \frac{1}{6}ms^2 + \frac{1}{2}ms^2 + 2m_1 s^2$$

$$I_{plate+bullet} = \frac{2}{3}ms^2 + 2m_1 s^2$$

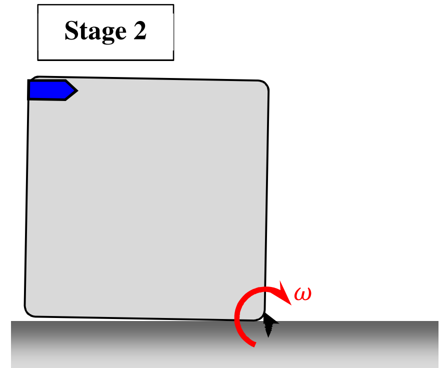
Plugging this into the bolded angular momentum relationship gives

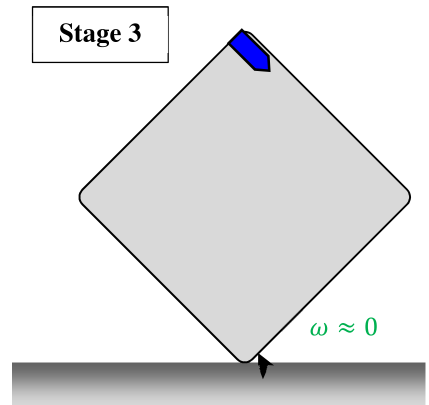
$$m_1 v s = \left(\frac{2}{3}ms^2 + 2m_1 s^2\right)\omega$$

Cancelling an s from each term gives

$$\boldsymbol{m_1 v = \left(\frac{2}{3}m + 2m_1\right)s\omega}$$

Notice: in this equation we know everything except m_1 & ω. Notice the units of each term are $\mathrm{kg}\cdot\frac{\mathrm{m}}{\mathrm{s}}$.
We can next compare stages 2 & 3 to get a second equation and (hopefully) use that equation to eliminate ω & solve for m_1…continues next page.





<u>Comparing stages 2 & 3</u> we see we have a single object (the combined bullet-plate object) in pure rotation. This is screaming out for conservation of energy. We are told there is negligible slipping of the block. We assume drag is negligible (unless otherwise specified). Therefore, we assume $W_{ext\ non-con} = 0$ giving

$$E_i = E_f$$

NOTE: we could get the center of mass height of the combined bullet+plate system or treat each mass separately when accounting for gravitational energy. Either way gets the same results.

$$m_1 g h_{1i} + m_2 g h_{2i} + \frac{1}{2} I_{total} \omega_i^2 = m_1 g h_{1f} + m_2 g h_{2f} + \frac{1}{2} I_{total} \omega_f^2$$

$$m_1 g(s) + mg\left(\frac{s}{2}\right) + \frac{1}{2}\left(\frac{2}{3} m s^2 + 2 m_1 s^2\right)\omega^2 = m_1 g\left(\sqrt{2} s\right) + mg\left(\frac{\sqrt{2}}{2} s\right) + 0$$

If you did the center of mass: initial center of mass height (above the ground) is

$$y_{CM} = \frac{s m_1 + \frac{s}{2} m}{m_1 + m}$$

The initial gravitational energy of the combined object is

$$(m_1 + m) g h_{CMi} = (m_1 + m) g \left(\frac{s m_1 + \frac{s}{2} m}{m_1 + m}\right) = m_1 g(s) + mg\left(\frac{s}{2}\right)$$

<u>Notice this gives the same result for the total initial gravitational potential energy.</u> A similar result holds for the final gravitational potential energy.

From here I would solve this equation for ω. Because ω is not given in the problem this will help eliminate it from the equations! First move the gravitational terms to one side and factor out as much as possible.

$$\frac{1}{2}\left(\frac{2}{3} m s^2 + 2 m_1 s^2\right)\omega^2 = \left(\sqrt{2} - 1\right) s g [m_1 + m]$$

Cancel an s from every term and isolate ω^2.

$$\boldsymbol{\omega^2 = \frac{2\left(\sqrt{2} - 1\right) g [m_1 + m]}{s\left(\frac{2}{3} m + 2 m_1\right)}}$$

Notice, on the right side, the units of mass and length will cancel giving the appropriate units of $\left(\frac{\text{rad}}{\text{s}}\right)^2$.

Compare this to the bold green equation from earlier. I first rearranged the green equation to solve for ω…

$$\boldsymbol{\omega = \frac{m_1 v}{\left(\frac{2}{3} m + 2 m_1\right) s}}$$

Square the green equation and set it equal to the previous equation.

$$\boldsymbol{\left(\frac{m_1 v}{\left(\frac{2}{3} m + 2 m_1\right) s}\right)^2 = \frac{2\left(\sqrt{2} - 1\right) g [m_1 + m]}{s\left(\frac{2}{3} m + 2 m_1\right)}}$$

$$\boldsymbol{\frac{m_1^2 v^2}{\left(\frac{2}{3} m + 2 m_1\right) s} = 2\left(\sqrt{2} - 1\right) g [m_1 + m]}$$

We see there is now only one unknown…our desired quantity m_1. Unfortunately, we see we have a quadratic formula to solve! First prepare the equation to be of the form $a m_1^2 + b m_1 + c = 0$.

Continues on the next page…

On the previous page we ended with

$$\frac{m_1^2 v^2}{\left(\frac{2}{3}m + 2m_1\right)s} = 2\left(\sqrt{2}-1\right)g[m_1 + m]$$

Multiply that ugly denominator over to the other side and use the FOIL method.

$$m_1^2 v^2 = 2\left(\sqrt{2}-1\right)g[m_1 + m]\left(\frac{2}{3}m + 2m_1\right)s$$

$$m_1^2 v^2 = 2\left(\sqrt{2}-1\right)gs\left[\frac{2}{3}mm_1 + \frac{2}{3}mm + 2m_1m_1 + 2m_1m\right]$$

I will choose to multiply all terms by 3. I will do this on the left side and *inside* the square brackets on the right side. I will also factor out a 2 from each term in the square brackets on the right hand side.
Finally, I will group the two m_1m terms inside the square brackets.

$$3m_1^2 v^2 = 4\left(\sqrt{2}-1\right)gs[4mm_1 + m^2 + 3m_1^2]$$

Notice: if I divide each side by $4\left(\sqrt{2}-1\right)gs$ I will collect all the ugly crap into a single term!!!

$$\frac{3v^2}{4\left(\sqrt{2}-1\right)gs}m_1^2 = 4mm_1 + m^2 + 3m_1^2$$

$$\left\{\frac{3v^2}{4\left(\sqrt{2}-1\right)gs} - 3\right\}m_1^2 - 4mm_1 - m^2 = 0$$

FINALLY we are ready for the quadratic formula with $a = \left\{\frac{3v^2}{4\left(\sqrt{2}-1\right)gs} - 3\right\}, b = -4m,\ \&\ c = -m^2$.
Note: recall, by order of operations, $-m^2 = -(m^2)$…the minus sign is NOT cancelled.

$$m_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m_1 = \frac{-(-4m) \pm \sqrt{(-4m)^2 - 4\left\{\frac{3v^2}{4\left(\sqrt{2}-1\right)gs} - 3\right\}(-m^2)}}{2\left\{\frac{3v^2}{4\left(\sqrt{2}-1\right)gs} - 3\right\}}$$

$$m_1 = \frac{4m \pm \sqrt{16m^2 + 4m^2\left\{\frac{3v^2}{4\left(\sqrt{2}-1\right)gs} - 3\right\}}}{2\left\{\frac{3v^2}{4\left(\sqrt{2}-1\right)gs} - 3\right\}}$$

If the denominator is *positive*, we must use the positive root (otherwise our result for m_1 is negative).
If the denominator is *negative*, the radical term must be smaller than $4m$. Regardless of our choice of root the answer for m_1 is negative (and makes no sense). This implies we require the positive root but also

$$\frac{3v^2}{4\left(\sqrt{2}-1\right)gs} - 3 > 0$$

$$\frac{3v^2}{4\left(\sqrt{2}-1\right)gs} > 3$$

$$v^2 > 4\left(\sqrt{2}-1\right)gs$$

$$v > 1.287\sqrt{gs}$$

$$\boldsymbol{m_1 = \left(1.333\frac{1 + \sqrt{0.250 + 0.4527\frac{v^2}{gs}}}{0.6036\frac{v^2}{gs} - 1}\right)m}$$