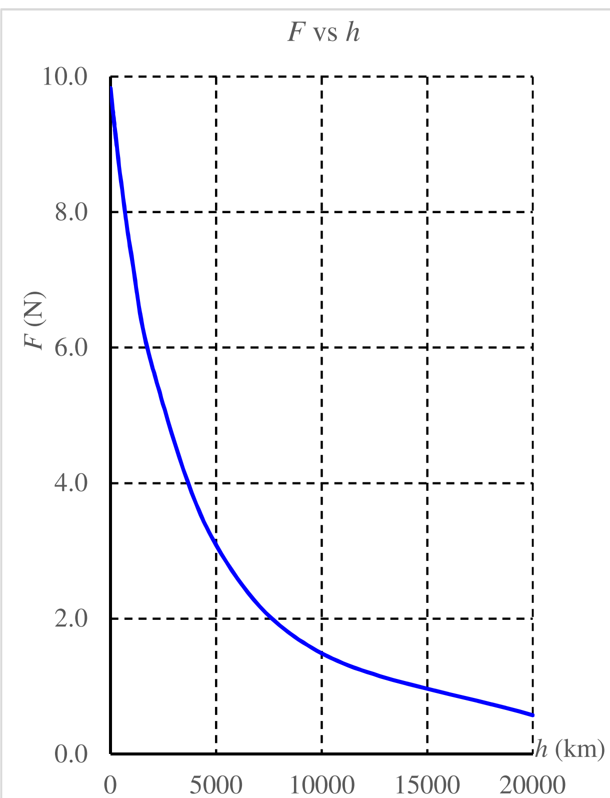
**Ch13**
**13.1**

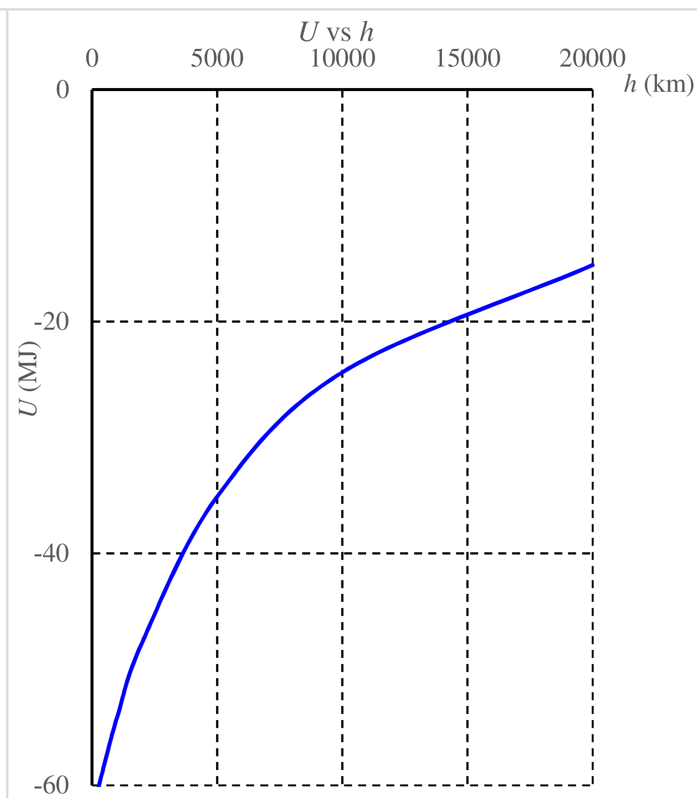
a) $3.53\times10^{22}$ N
b) $1.99\times10^{20}$ N
c) $-5.29\times10^{33}$ J…make sure you included the negative sign!
d) $-7.63\times10^{28}$ J
e) The potential energy of the system is the potential energy of all pairs of masses. Adding the previous two results neglects the sun-moon pair with $-6.52 \times 10^{31}$ J. The total potential energy of the three mass system is $U_{tot} = U_{12} + U_{13} + U_{23} = -5.36 \times 10^{33}$ J.
f) The moon is about 400 times closer but exerts about 180 times less force. Notice that the radius of the earth is negligible compared to the earth-sun distance but not compared to the earth-moon distance. To a first approximation, the gravitational field from the sun is constant across the entire earth while the gravitational field from the moon changes significantly from one side of the earth to the other. This is why the moon is the dominant influence on the tides.

**13.2** Table and plots are shown. Watch those units!!!

Most people consider weight the force of gravity exerted by the earth. Notice even in space the 1 kg mass is not weightless. People experience the feeling of weightlessness whenever they are in freefall. Objects in orbit about the earth are in freefall. This is why astronauts in orbit feel weightless even though the force of weight is acting on them.

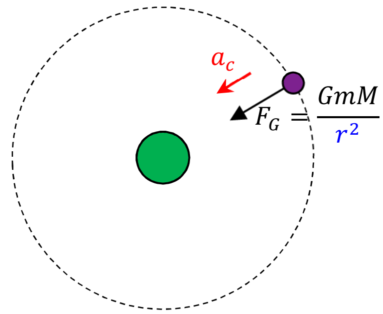
| $h$ (km) | $F$ (N) | $U$ (MJ) |
|---|---|---|
| 0 | 9.8 | -63 |
| 100 | 9.5 | -62 |
| 200 | 9.2 | -61 |
| 500 | 8.5 | -58 |
| 1000 | 7.3 | -54 |
| 2000 | 5.7 | -48 |
| 5000 | 3.1 | -35 |
| 10000 | 1.5 | -24 |
| 20000 | 0.6 | -15 |



**13.3**

a) $U = -\frac{GmM}{r} = -\frac{GmM}{R+h}$




b) The FBD for this problem occasionally confuses people as they are expecting more forces. Only one force, gravity, acts on the satellite in orbit. The satellite is in free fall. Solving the force equation for $v$ gives

$$v = \sqrt{\frac{GM}{R+h}}$$

Notice the mass of the satellite does not affect the speed! All satellites at the same altitude will orbit with the same speed!

c) For a circular orbit $K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GmM}{R+h}$.

Notice $K = -\frac{1}{2}U$.

If you are really into strange/mathy web searches, consider searching "virial theorem".

d) For a circular orbit $E = K + U = -\frac{1}{2}U + U = \frac{1}{2}U = -\frac{1}{2}\frac{GmM}{R+h}$.

Notice the total energy is negative.
I like to think about it this way: if the energy is zero the planet is having no effect on you.
If the energy is negative, you don't have enough energy to escape and can be stuck in orbit.
It's like you are in debt to a bank too big to fail & are forced to work your whole life to pay the man…well…kinda.
Notice as you go to a higher orbit the value of $E$ is *less* negative.
It can be confusing to say it has more potential energy since $E$.
I sometimes prefer to say "more positive" or "less negative".
Just like moving -3 to the left equals moving +3 to the right, less negative is the same as more positive.

e) Finally, as you go to higher orbit the *change* in potential energy is positive.
This is true for both $U = mgh$ and $U = -\frac{GmM}{R+h}$.
Hopefully this makes negative energy sit with you a little better.
In problem **13.12** I address how $mgh$ and $-GmM/r$ are related to each other…

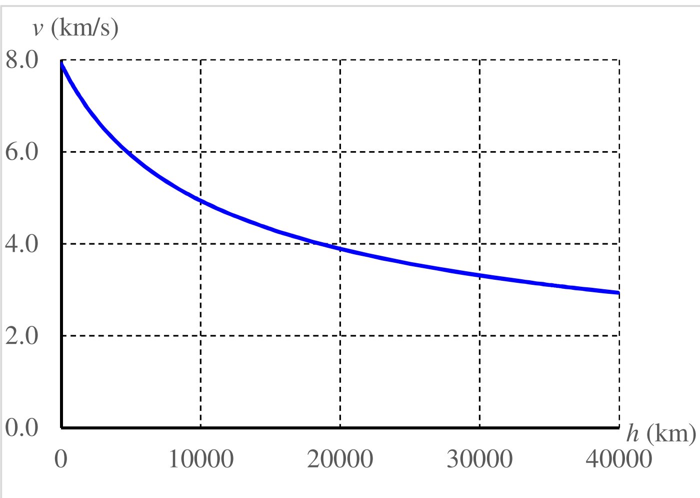
f) For a circular orbit $v = \frac{distance}{time} = \frac{2\pi(R+h)}{\mathbb{T}}$ which gives

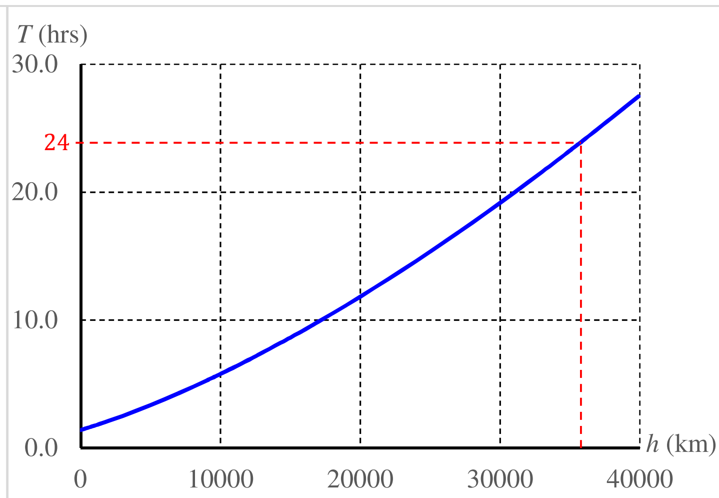
$$\mathbb{T} = 2\pi\sqrt{\frac{(R+h)^3}{GM}} = some\ constant \times (orbit\ radius)^{3/2}$$

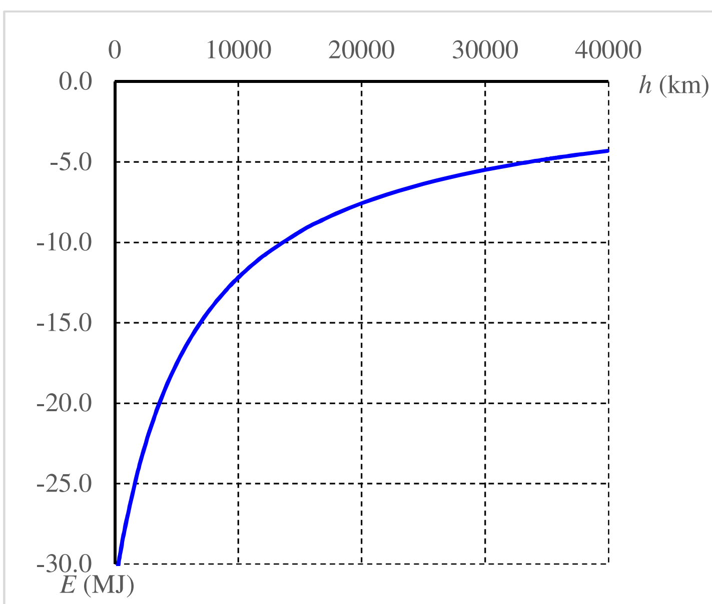
g) Table and plots are shown on the next page.
**Watch those units.**
Tip: make more columns of data than what I displayed.
Do your work/calculations in the spreadsheet in scientific notation.
Then, just before making the plots, make columns of data with appropriate prefixes so they display nicely.
Notice the special period at approximately 36000 km…hmmmm

| $M$ (kg) | $m$ (kg) | $R$ (km) |
|---|---|---|
| 5.98E+24 | 1.00 | 6370 |

| $h$ (km) | $v$ (km/s) | $T$ (hr) | $E$ (MJ) |
|---|---|---|---|
| 0 | 7.9 | 1.4 | -31.3 |
| 1000 | 7.4 | 1.7 | -27.1 |
| 2000 | 6.9 | 2.1 | -23.8 |
| 5000 | 5.9 | 3.3 | -17.5 |
| 10000 | 4.9 | 5.8 | -12.2 |
| 15000 | 4.3 | 8.6 | -9.3 |
| 20000 | 3.9 | 11.8 | -7.6 |
| 25000 | 3.6 | 15.4 | -6.4 |
| 30000 | 3.3 | 19.2 | -5.5 |
| 35000 | 3.1 | 23.2 | -4.8 |
| **35900** | **3.1** | **24.0** | **-4.7** |
| 40000 | 2.9 | 27.6 | -4.3 |



**13.3½** The satellite is not held up at all!  The satellite is constantly falling towards the earth.  The satellite is moving very rapidly.  The surface of the earth curves away beneath the satellite before it is able to impact the surface.  Look at the FBD; there is only one force.  Gravity acts towards the center of the circle.  There is no outwards force "holding up" the satellite!

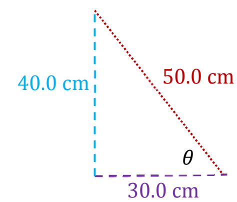
**13.4** Use a simulation to check your work.  If you are wondering how you could *calculate* the forces, don't worry.  Most of the simulation questions are patterned after the problems that follow.  You will determine algebraic expressions for the forces in those problems.

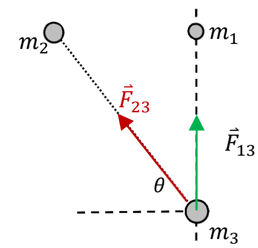
**13.5**

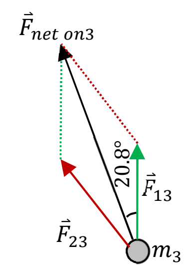
a) The force from $m_2$ has double the mass but about 67% greater radius. The radius is squared. Expect $F_{2on3}$ is about 40% larger than $F_{2on3}$. Should point up and between 1 & 2, a little closer to 2 than 1.

b) Style 1 splits into components using trig while Style 2 uses unit vectors to split into components. For 3D problems, style 2 is required. Did you remember to convert cm to m? Notice how using a scientific prefix makes the work neat…

| **Style 1** | **Style 2** |
|---|---|
| Use 1 nN = $10^{-9}$ N<br>$F_{13} = 1.25\text{ nN}$<br>$F_{23} = 1.600\text{ nN}$<br><br>$\vec{F}_{13} = 0\,\hat{\imath} + 1.25\hat{\jmath}$<br>$\vec{F}_{23} = -1.600\cos\theta\,\hat{\imath} + 1.600\sin\theta\,\hat{\jmath}$<br><br>$\cos\theta = \frac{30.0\text{cm}}{50.0\text{cm}} = 0.6$<br>$\sin\theta = \frac{40.0\text{cm}}{50.0\text{cm}} = 0.8$<br><br>$\vec{F}_{net\ on3} = -0.96\text{ nN}\,\hat{\imath} + 2.53\text{ nN}\,\hat{\jmath}$<br><br>$\vec{F}_{net\ on3} = 2.71\text{ nN @ }20.8°\text{ W of N}$ | $F_{13} = 1.25\text{ nN}$<br>$F_{23} = 1.600\text{ nN}$<br><br>$\hat{r}_{\mathbf{1to3}} = \frac{-40.0\text{cm}\hat{\jmath}}{40.0\text{cm}} = -\hat{\jmath}$<br><br>$\hat{r}_{\mathbf{2to3}} = \frac{30.0\text{cm}\hat{\imath} - 40.0\text{cm}\hat{\jmath}}{50.0\text{cm}}$<br>$\hat{r}_{23} = 0.600\hat{\imath} - 0.800\text{cm}\hat{\jmath}$<br><br>$\vec{F}_{13} = F_{13}(-\hat{r}_{13})$<br>$\vec{F}_{23} = F_{23}(-\hat{r}_{23})$<br><br>$\vec{F}_{net\ on3} = 2.71\text{ nN @ }20.8°\text{ W of N}$ |




**WATCH OUT!** Many times students make the following mistake:

$$\vec{F}_{23} \neq -\frac{Gm_1m_2}{x^2}\hat{\imath} + \frac{Gm_1m_2}{y^2}\hat{\jmath}$$

The denominator always uses the total distance between the masses (never the *x* or *y* distance).

c) The earth exerts a gravitational force of $m_3 g \approx 30\text{N}$ on $m_3$.
This is about 10 billion times greater!
It can be interesting to read about the Cavendish (1797) and Schiehallion (1774) experiments to see how physicists were able to measure such small forces and to estimate the density of the earth.
Amazing!

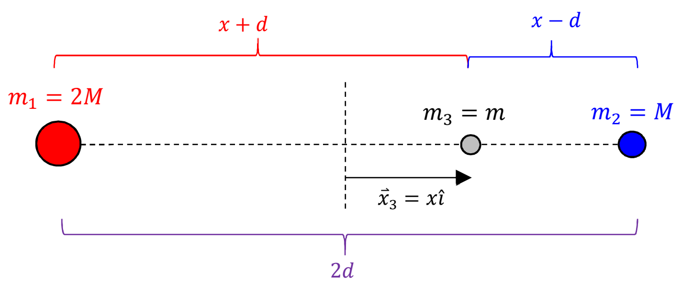
d) $U_{tot} = U_{12} + U_{13} + U_{23} = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}} = -G\left(\frac{2.00}{0.300} + \frac{3.00}{0.400} + \frac{6.00}{0.500}\right) = -1.7\underline{4}5\text{ nJ}$

**13.6**
**Part a on this page:**
At first it may seem easiest to do this entire problem using *scalar distances*.
After years of teaching this, I think students actually find this more understandable if we use *vector positions*.
By doing this, we remove the burden of dealing with three regions of interest on a case by case basis.

If you attack this problem using *distances* (instead of *positions*), you must consider three regions:

- **Region 1:** Little $m$ to the *left* of masses $2M$ & $M$ $(x < -d)$
- **Region 2:** Little $m$ *between* masses $2M$ & $M$ $(-d < x < d)$
- **Region 3:** Little $m$ to the *right* of masses $2M$ & $M$ $(x > d)$

First let us describe an arbitrary *position* the first quadrant (in this case, an arbitrary positive value of $x$).




The *positions* of each mass are:

$$\vec{r}_1 = -d\hat{\imath}$$
$$\vec{r}_2 = d\hat{\imath}$$
$$\vec{r}_3 = x\hat{\imath}$$

The *distances* between masses are:

$$r_{1to2} = \|\vec{r}_2 - \vec{r}_1\| = \|d\hat{\imath} - (-d\hat{\imath})\| = 2d$$
$$r_{1to3} = \|\vec{r}_3 - \vec{r}_1\| = \|x\hat{\imath} - (-d\hat{\imath})\| = |x + d|$$
$$r_{2to3} = \|\vec{r}_3 - \vec{r}_2\| = \|x\hat{\imath} - d\hat{\imath}\| = |x - d|$$

Note: I
Notice: we still get correct distances between masses regardless of the value (+ or -) used for $x$.
If you don't believe me, try a few cases with numbers ( set $d = 3$ and try $x = \pm 2$ & $x = \pm 5$).

Now attempt the energy problem.

$$U = U_{12} + U_{13} + U_{23}$$

$$U = -\frac{G(2M)(M)}{2d} - \frac{G(2M)(m)}{|x + d|} - \frac{G(M)(m)}{|x - d|}$$

$$U = -GM\left(\frac{M}{d} + \frac{2m}{|x + d|} + \frac{m}{|x - d|}\right)$$

The following form is useful if one wants to take derivatives.

$$U = -GM\left(\frac{M}{d} + \frac{2m}{\sqrt{(x + d)^2}} + \frac{m}{\sqrt{(x - d)^2}}\right)$$

**Part b: I discuss an alternate style using $\boldsymbol{F_x = -\frac{dU}{dx}}$ at the bottom of this page…**

At first I wrote $\vec{F}_{net} = F_x\hat{\imath}$ where $F_x = GMm\left[\frac{1}{(d-x)^2} + \frac{-2}{(d+x)^2}\right]$ and assumed this should always work. After making the plots and doing problem **13.19** I realized I made a mistake! When $x < -d$ both forces are to the right which flips the sign on the second term. When $x > d$ both masses pull to the left which flips the sign of the first term instead!

The *positions* of each mass are:

$$\vec{r}_1 = -d\hat{\imath}$$
$$\vec{r}_2 = d\hat{\imath}$$
$$\vec{r}_3 = x\hat{\imath}$$

The *displacements* between masses are:

$$\vec{r}_{1to3} = \vec{r}_3 - \vec{r}_1 = x\hat{\imath} - (-d\hat{\imath}) = (x+d)\hat{\imath}$$
$$\vec{r}_{2to3} = \vec{r}_3 - \vec{r}_2 = x\hat{\imath} - d\hat{\imath} = (x-d)\hat{\imath}$$

Again, we already found the *distances* between masses are:

$$r_{1to3} = \|\vec{r}_3 - \vec{r}_1\| = \|x\hat{\imath} - (-d\hat{\imath})\| = |x+d|$$
$$r_{2to3} = \|\vec{r}_3 - \vec{r}_2\| = \|x\hat{\imath} - d\hat{\imath}\| = |x-d|$$

Now it is straightforward to determine the forces:

$$\vec{F}_{NETon3} = \vec{F}_{1on3} + \vec{F}_{2on3}$$
$$\vec{F}_{NETon3} = \left(-\frac{Gm_1m_3}{r_{1to3}^2}\hat{r}_{1to3}\right) + \left(-\frac{Gm_2m_3}{r_{2to3}^2}\hat{r}_{2to3}\right)$$
$$\vec{F}_{NETon3} = \left(-\frac{Gm_1m_3}{\boldsymbol{r_{1to3}^3}}\vec{\boldsymbol{r}}_{\boldsymbol{1to3}}\right) + \left(-\frac{Gm_2m_3}{\boldsymbol{r_{2to3}^3}}\vec{\boldsymbol{r}}_{\boldsymbol{2to3}}\right)$$
$$\vec{F}_{NETon3} = \left(-\frac{G(2M)m}{|x+d|^3}(x+d)\hat{\imath}\right) + \left(-\frac{G(M)m}{|x-d|^3}(x-d)\hat{\imath}\right)$$
$$\boldsymbol{\vec{F}_{NETon3} = -GMm\left(2\frac{(x+d)\hat{\imath}}{|x+d|^3} + \frac{(x-d)\hat{\imath}}{|x-d|^3}\right)}$$

**Alternate style using $\boldsymbol{F_x = -\frac{dU}{dx}}$**

$$F_x = -\frac{dU}{dx}$$
$$F_x = -\frac{d}{dx}GM\left(\frac{M}{d} + \frac{2m}{\sqrt{(x+d)^2}} + \frac{m}{\sqrt{(x-d)^2}}\right)$$

Notice $\frac{d}{dx}\frac{M}{d} = 0…$

$$F_x = -GM\frac{d}{dx}\{2m[(x+d)^2]^{-1/2} + m[(x-d)^2]^{-1/2}\}$$
$$F_x = -GM\left\{2m\frac{d}{dx}[(x+d)^2]^{-1/2} + m\frac{d}{dx}[(x-d)^2]^{-1/2}\right\}$$

From there one could use the chain rule or plug it into a symbolic derivative program online….

**Solution continues on the next page…**

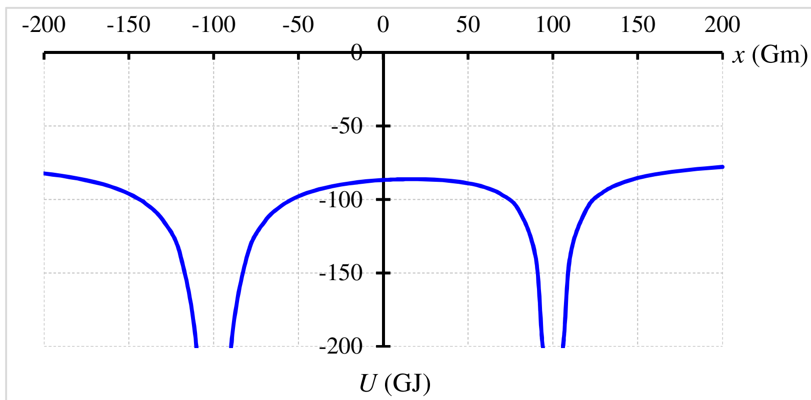
c) In equilibrium $\vec{F}_{net} = 0$. Setting part b equal to zero and one finds $x = 0.172d$.
Notice the equilibrium point is not at the center of mass for $M$ and $2M$.
For reference the center of mass position for $M$ and $2M$ is $x_{\text{CM}} = \frac{(-d)2M+(d)M}{3M} = -\frac{1}{6}d$.
While not exactly the same, this problem reminds me of the Roche limit where a moon orbiting a planet is ripped apart and turns into rings like Saturn's rings. Do a web search…

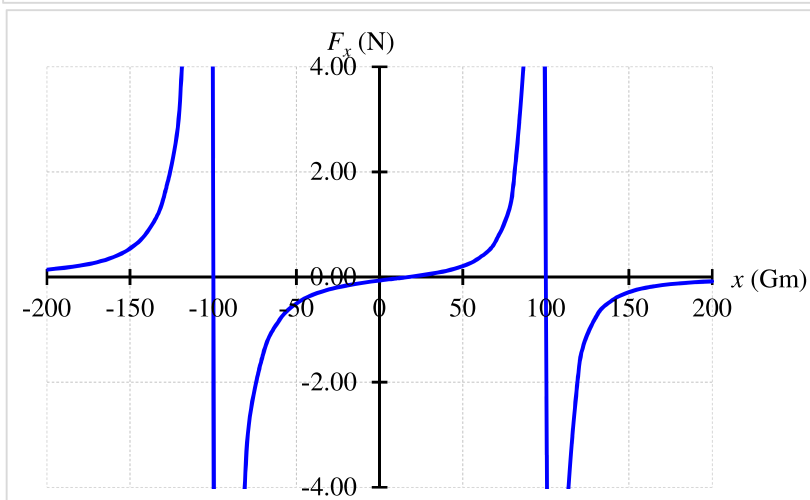
d) The plot (shown below) can be tricky since there are asymptotes at $x = \pm d$.
The defect at $x = \pm d$ stems from force equation skipping from negative infinity to positive infinity.
For the most part I used increments of 0.1E11 for $x$.
I skipped $x = \pm d = \pm 1\text{E}11$ and instead used $x = d \pm 0.05\text{E}11$ and $x = -d \pm 0.05\text{E}11$ (see table below).
Notice the data table is much easier to read using the prefix of Gm = $10^9$ m for $x$.
The equilibrium point is at $x = 0.172d$ as expected.

e) Plot shown below. If we are to allow $x = -2d$ to $2d$ we need the absolute value in the denominator.
Again, notice how clean the data looks if the appropriate prefix is used as opposed to sticking with scientific notation. Usually the extra headache of dealing with unit/prefix conversions in your data is well worth the effort when you try to actually communicate with someone.
**You might recall chapter 8 when we discussed $F_x = -\frac{dU}{dx}$.**
**For practice you can compare the negative slope of $U$ vs $x$ to $F_x$.**

| *d* (m) | *M* (kg) | *m* (kg) | *d* (Gm) |
|---|---|---|---|
| 1.00E+11 | 1.00E+16 | 1.00E+15 | 100 |

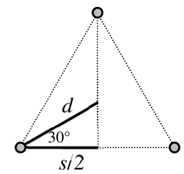
| *x* (m) | *x* (Gm) | *F* (N) | *U* (GJ) |
|---|---|---|---|
| -2.00E+11 | -200 | 0.14 | -82 |
| -1.90E+11 | -190 | 0.17 | -84 |
| … | … | … | … |
| -1.10E+11 | -110 | 13.36 | -203 |
| -1.05E+11 | -105 | 53.38 | -337 |
| -9.50E+10 | -95 | -53.34 | -337 |
| -9.00E+10 | -90 | -13.32 | -204 |
| -8.00E+10 | -80 | -3.31 | -137 |
| … | … | … | … |
| 9.00E+10 | 90 | 6.63 | -140 |
| 9.50E+10 | 95 | 26.64 | -207 |
| 1.05E+11 | 105 | -26.71 | -207 |
| 1.10E+11 | 110 | -6.70 | -140 |
| 1.20E+11 | 120 | -1.70 | -106 |
| … | … | … | … |
| 2.00E+11 | 200 | -0.08 | -78 |



**13.7**

a) $h = s \sin 60° = \frac{s}{2} \tan 60° = \frac{\sqrt{3}}{2} s \approx 0.8660s$




b) $d = \frac{s}{2\cos 30°} = \frac{s}{\sqrt{3}} \approx 0.5774s$. If you are having trouble, consider the figure at right.

c) After all that geometry, turns out this one is a breeze!
There are three possible pairs but each in each pair the masses are separated by distance $s$.
One finds

$$U_i = U_{12} + U_{13} + U_{23} = -3\frac{Gm^2}{s}$$

d) Adding in a mass at the center will add three pairs making

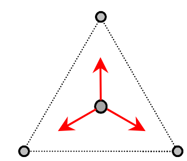
$$U_f = U_i + U_{41} + U_{42} + U_{43} = -3\frac{Gm^2}{s} - 3\frac{Gm^2}{d}$$

**Watch out…**this problem did NOT ask for the potential energy when a fourth mass was added.
It asked for the CHANGE in potential energy when the fourth mass was added.
The change in potential energy is given by

$$\Delta U = U_f - U_i = -3\frac{Gm^2}{d} = -3\sqrt{3}\frac{Gm^2}{s} = -5.196\frac{Gm^2}{s}$$

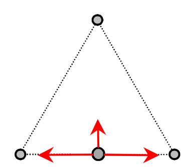
e) Work done by gravity is determined using $W = -\Delta U = 5.196\frac{Gm^2}{s}$.
Just like a ball falling to earth, gravity does positive work when the potential energy becomes more negative (or less positive).

f) Think about the symmetry of the problem (figure at right).
All the forces balance at the center! $F_{net} = 0$.



g) If the fourth mass is halfway between 1 & 3, the forces from 1 and 3 balance.
Consider the figure at right.
Therefore

$$\vec{F}_{net} = \vec{F}_{2on4} = \frac{Gm^2}{h^2}\hat{j} = \frac{4Gm^2}{3s^2}\hat{j} = 1.333\frac{Gm^2}{s^2}\hat{j}$$



**13.8**

a) $\vec{F}_{14} = \frac{Gm^2}{2s^2}\left(-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right) = \frac{Gm^2}{2\sqrt{2}s^2}(-\hat{\imath} + \hat{\jmath})$ while $\vec{F}_{24} = \frac{Gm^2}{s^2}\hat{\jmath}$ and $\vec{F}_{34} = -\frac{Gm^2}{s^2}\hat{\imath}$.

Vector math gives $F_{net} = 1.914\frac{Gm^2}{s^2}$ directed towards 1 (northwest).

b) $U = U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34} = -4\frac{Gm^2}{s} - 2\frac{Gm^2}{\sqrt{2}s} = -5.414\frac{Gm^2}{s}$

c) At the center $\vec{F}_{24} = -\vec{F}_{34}$; $\vec{F}_{24}$ & $\vec{F}_{34}$ cancel by symmetry!

Therefore $\vec{F}_{net} = \vec{F}_{14} = \frac{2Gm^2}{\sqrt{2}s^2}(-\hat{\imath} + \hat{\jmath})$ or $F_{net} = \frac{2Gm^2}{s^2}$ directed NW.

It is nearly the same (4% difference) but not exactly equal to part a.

d) I think about it like this: when you drop a block towards the earth the change in GPE is negative (the GPE becomes less positive). In this case, mass 4 gets closer to the rest of the mass so the change in GPE should be negative (the GPE becomes more negative). Notice that less positive and more negative mean the same thing.

$$U_f = -2\frac{Gm^2}{s} - \frac{Gm^2}{\sqrt{2}s} - 3\frac{Gm^2}{\left(\frac{s}{\sqrt{2}}\right)} = -6.950\frac{Gm^2}{s}$$

The potential energy becomes more negative. The <u>change</u> in potential energy is negative.

e) Recall $Work = -\Delta U$. Gravity does positive work on mass 4 as it accelerates from rest.

f) The symmetry of the problem is such that the combination of $\vec{F}_{24} + \vec{F}_{34}$ will always point towards mass 1. The net force will always point towards mass 1 and thus the object will always accelerate towards mass 1.

g) The force, and thus acceleration, are not constant. Don't use kinematics of constant acceleration! From conservation of energy we know $K_i + U_i = K_f + U_f$ or $\Delta K = -\Delta U$.

Since $K_i = 0$ this gives $K_f = U_i - U_f$. Furthermore we know $K_f = \frac{1}{2}mv^2$.

Plugging in and solving for $v$ gives

$$v = \sqrt{\frac{2(U_i - U_f)}{m}} = \sqrt{\frac{-2\Delta U}{m}} = \sqrt{\frac{2|\Delta U|}{m}} = \sqrt{\frac{2}{m}\left(-5.414\frac{Gm^2}{s} + 6.950\frac{Gm^2}{s}\right)} = 1.75\sqrt{\frac{Gm}{s}}$$

If mass 1 is $2m$, parts a, b, d, and g will <u>not</u> double. Part c <u>does</u> double since $\vec{F}_{24}$ & $\vec{F}_{34}$ cancel by symmetry at the center! Parts e and f won't change! Part a becomes $F_{net} = 1.914\frac{Gm^2}{s^2}$ directed NW. Part b becomes $U = -\frac{Gm^2}{s}\left(6 + \frac{3}{\sqrt{2}}\right) = -8.121\frac{Gm^2}{s}$. Part d becomes $U = -\frac{Gm^2}{s}\left(4 + 4\sqrt{2} + \frac{1}{\sqrt{2}}\right) = -10.36\frac{Gm^2}{s}$. Part g becomes $v = 2.12\sqrt{\frac{Gm}{s}}$.

I was curious so I reworked the problem using an acceleration obtained from part a.

h) As long as the forces on 4 are symmetric about diagonal between 1 & 4 we know mass 4 will travel directly towards 1. If mass 1 or 4 is doubled, the forces are still symmetrical on 4. If mass 2 or 3 is doubled the symmetry is broken and 4 no longer travels directly towards 1. The particle orbits in an unusual non-circular way. If both 2 & 3 are doubled or both 1 & 4 are doubled the symmetry still holds. To observe what happens when the symmetry is broken, I opened the electric field hockey PHeT and created a square with three negative charges and the fourth being positive. I doubled one of the negative charges adjacent to the positive mass and hit play. Watching the motion was pretty cool!

**13.9**

a) Using conservation of energy

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv^2 - \frac{GmM}{R} = \frac{1}{2}mv_f^2 - \frac{GmM}{R+h}$$

$$\frac{1}{2}mv^2 - \frac{GmM}{R} \approx 0 - \frac{GmM}{R+\infty} = 0$$

$$v = v_{escape} = \sqrt{\frac{2GM}{R}}$$

b) 11.2 km/s

c) If the muzzle speed of the bullet is less than $v_{escape}$ it will have less initial kinetic energy.
This implies it will reach a lower altitude and ultimately return to earth.
It will return to earth at terminal speed but since bullets are aerodynamic it will still have enough velocity to tear a hole in a roof or kill someone.
Not a good idea.

d) The earth is rotating once per day giving $\omega = 72.7\ \frac{\mu\text{rad}}{\text{s}}$.
An object on earth is thus rotating with radius $r = R\cos\theta$.
The closer to the equator, the lower the latitude (angle $\theta$).
The initial speed of an object is not zero but $v = r\omega = R\omega\cos\theta$.
One launch site in Florida has $\theta = 28.4°$ giving an initial speed of $v = 407\ \text{m/s}$ before any fuel is used!
By comparison, launching at the north pole the initial speed is 0 prior to use of fuel.
Launching closer to the equator increases initial speed and should lower fuel costs.

e) Working it out

$$R = \frac{2GM}{c^2} = 0.0088\ \text{m} \approx 1\ \text{cm}$$

This radius is called the event horizon or Schwarzschild radius.
For a black hole with mass equal to the earth you find

$$R = 0.0088\ \text{m} \approx 1\ \text{cm}$$

Any object with mass *M* and radius *smaller* than the Schwarzschild radius will be a black hole.

Note: black holes which *are not* rotating are called Schwarzschild black holes.
Black holes which *are* rotating are called Kerr black holes.
If the black holes carry net charge, evidently the names (and properties) change again.

**13.10** **I assumed the planet was earth with mass $M = 5.97 \times 10^{24}$ kg & radius $R = 6370$ km.**

a) Gravity is a conservative force. Since the only external force is conservative, we may use conservation of energy as long as we include the potential energy associated with that force.

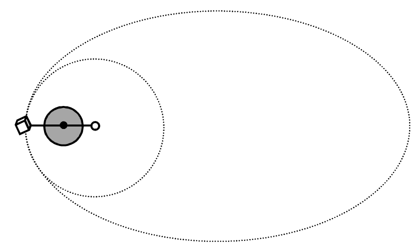
b) It seems logical to call the center of the planet the origin. If this is done, the line of action for the gravitational force always runs through the pivot point (the origin) and will cause no external torque. If there is no external torque, angular momentum is conserved.

c) To get the radius at **B** I used the Pythagorean theorem. To get the force at **B** and **C** I used ratios. For example, **B** has 5 times the radius so it has $\frac{1}{5^2}$ times the force. From Chapter 11, recall that $r_\perp$ is the shortest distance from the line of motion through $\vec{v}$ to the origin. For a point particle we know $L = mvr_\perp$. From conservation of angular momentum we know

$$L_A = L_B$$
$$mv_A r_{\perp A} = mv_B r_{\perp B}$$
$$v_B = v_A \frac{r_{\perp A}}{r_{\perp B}}$$

For the last three columns use $K = \frac{1}{2}mv^2$, $U = -\frac{GmM}{r}$, and $E = K + U$. As we expect both angular momentum and energy are conserved.

Note: to determine the first row I looked up the radius of curvature of an ellipse.



The radius of curvature at point **A** (from CubeSat to white dot) is $\frac{b^2}{a} = 3.6R$.

Force depends on the center-to-center distance $2R$ (to black dot).

This gave a force equation

$$m\frac{v_A^2}{3.6R} = \frac{GmM}{(2R)^2}$$

Solving for speed gives

$$v_A = \sqrt{\frac{0.9GM}{R}} = 7.51\ \frac{\text{km}}{\text{s}}$$

In elliptical orbits we can no longer assume $K = -\frac{1}{2}U$.

Also notice at points **B** and **D** the force does *not* act towards the center of circular motion causing tangential acceleration.

To use circular motion FBDs at **B** or **D** use the radius of curvature given by $\frac{a^2}{b} = 16.7R$.

Don't forget to split up the force into radial and tangential components.

I found $m\frac{v_B^2}{16.7R} = \frac{GmM}{(10R)^2}0.6$ where the 0.6 comes from splitting into components.

This yields

$$v_B = \sqrt{\frac{0.1GM}{R}} = 2.50\ \frac{\text{km}}{\text{s}}$$

| **Point** | $r$ (units of $R$) | $F$ (N) | $r_\perp$ (units of $R$) | $L$ (kg·km²/s) | $v$ (km/s) | $K$ (MJ) | $U$ (MJ) | $E$ (MJ) |
|---|---|---|---|---|---|---|---|---|
| **A** | 2 | 2.45 | 2 | 95.7 | 7.51 | 28.2 | -31.3 | -3.1 |
| **B** | 10 | 0.098 | 6 | 95.7 | 2.50 | 3.1 | -6.3 | -3.1 |
| **C** | 18 | 0.030 | 18 | 95.7 | 0.83 | 0.3 | -3.5 | -3.1 |

In a course called classical mechanics, should you ever take it, you will learn how to handle elliptical orbits using a concept know as eccentricity.

**13.11**

a) $\Delta U = -\int_i^f \frac{GMm}{r^2}(-\hat{r})\cdot dr\hat{r} = \int_i^f \frac{GMm}{r^2}dr = \left[-\frac{GMm}{r}\right]_i^f = -\left[\frac{GMm}{r_f} - \frac{GMm}{r_i}\right] = \frac{GMm}{r_i} - \frac{GMm}{r_f}$

b) $\Delta U = \frac{GMm}{R} - \frac{GMm}{\infty} = \frac{GMm}{R}$.

We also know that $\Delta U = U_f - U_i = U_{r=\infty} - U_{r=R}$.

Since we expect $U_{r=\infty} = 0$ one finds $U = -\frac{GMm}{R}$.

This is how you derive the result for the gravitational energy stored in a pair of point masses.

**13.12** Solutions mentioned in problem statement.

**13.13** Note: this problem uses all lower case $m$'s as opposed to **13.6** and **13.19**. Also, to keep things easy, I am assuming $x$ is always between the two masses so I don't need to worry about absolute value signs.

a) $U = -Gm^2\left(\frac{2}{x} + \frac{2}{r} + \frac{1}{r-x}\right)$

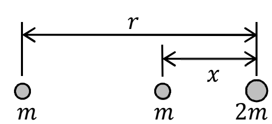
b) $F_x = \frac{Gm^2}{(r-x)^2} - \frac{2Gm^2}{x^2}$

c) You should find that $F_x = -\frac{dU}{dx}$

d) To find the force on the rightmost mass use $F_r = -\frac{dU}{dr}$

e) To use this trick, I typically set up the problem so the mass of interest is on the right side of the coordinate system to reduce minus sign errors. In the previous two calculations we are essentially stating that $2m$ is at the origin and one of the other masses is fixed.

To determine the force on $2m$ I would flip the picture like the one shown at right.




Then use $F_r = -\frac{dU}{dr}$. Here comes another annoying part: changing $r$ will affect $x$!

If you lengthen $r$ by 1 cm you also lengthen $x$ by 1 cm.

Therefore $dr = dx$ or $\frac{dx}{dr} = 1$.

You will need to use the chain rule on terms with $x$ since it is no longer constant!

Note: one could define the constant distance $L = r - x$.

If the equation for $U$ is written in terms of $r$ and $L$ no chain rule is required since $\frac{dL}{dr} = 0$.

After all this, don't forget we started with a picture that was flipped the other way.

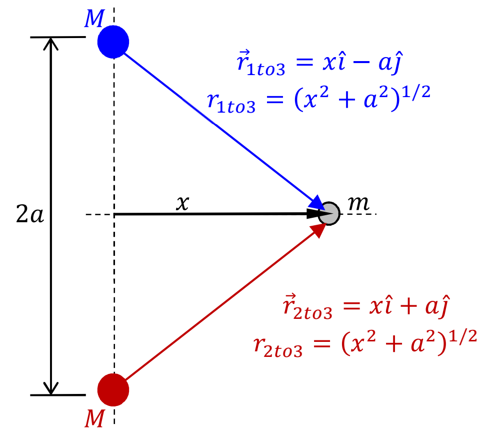
To account for this simply multiply your final answer by $-1$.

**In this case it would have been easier to start from forces directly.**

**13.14**

a) $U = -G\left(\frac{M^2}{2a} + \frac{2Mm}{(a^2+x^2)^{1/2}}\right)$

b) See the figure and the work below.




$$\vec{F}_{NET} = \vec{F}_{1on3} + \vec{F}_{2on3}$$

$$\vec{F}_{NET} = \frac{GMm}{r_{1to3}^2}(-\hat{r}_{1to3}) + \frac{GMm}{r_{2to3}^2}(-\hat{r}_{2to3})$$

$$\vec{F}_{NET} = \frac{GMm}{r_{1to3}^3}(-\vec{r}_{1to3}) + \frac{GMm}{r_{2to3}^3}(-\vec{r}_{2to3})$$

$$\vec{F}_{NET} = \frac{GMm}{(x^2+a^2)^{3/2}}\big(-(x\hat{\imath} - a\hat{\jmath})\big) + \frac{GMm}{(x^2+a^2)^{3/2}}\big(-(x\hat{\imath} + a\hat{\jmath})\big)$$

$$\vec{F}_{NET} = \frac{-GMm}{(x^2+a^2)^{3/2}}\big((x\hat{\imath} - a\hat{\jmath}) + (x\hat{\imath} + a\hat{\jmath})\big)$$

$$\vec{F}_{NET} = \frac{-GMm}{(x^2+a^2)^{3/2}}(2x\hat{\imath})$$

$$\boldsymbol{\vec{F}_{NET} = -\frac{2GMmx}{(x^2+a^2)^{3/2}}\hat{\imath}}$$

c) Take the derivative with respect to $x$ and set equal to zero.
Solve for $x$ which gives the position of the maximum.
Note: rather than use the quotient rule, I flip the denominator to the top and make the exponent negative.

$$F_x = -2GmMx(a^2+x^2)^{-3/2}$$

$$\frac{d}{dx}F_x = 0$$

Note: the constants factor out…then drop out.
When you divide each side by the constant, zero divided by the constant is still zero.

$$\frac{d}{dx}x(a^2+x^2)^{-3/2} = 0$$

$$(a^2+x^2)^{-3/2}\frac{\boldsymbol{d}}{\boldsymbol{dx}}\boldsymbol{x} + x\frac{d}{dx}(a^2+x^2)^{-3/2} = 0$$

$$(a^2+x^2)^{-3/2}(\mathbf{1}) + x\left(-\frac{3}{2}\right)(a^2+x^2)^{-5/2}\left(\frac{d}{dx}(a^2+x^2)\right) = 0$$

$$(a^2+x^2)^{-3/2} + x\left(-\frac{3}{2}\right)(a^2+x^2)^{-5/2}(2x) = 0$$

$$(a^2+x^2)^{-3/2} - 3x^2(a^2+x^2)^{-5/2} = 0$$

$$(a^2+x^2)^{-3/2} = 3x^2(a^2+x^2)^{-5/2}$$

Multiply each side by $(a^2+x^2)^{+5/2}$.

$$(a^2+x^2)^{-3/2}(a^2+x^2)^{+5/2} = 3x^2$$

$$(a^2+x^2)^{+2/2} = 3x^2$$

$$a^2 + x^2 = 3x^2$$

$$a^2 = 2x^2$$

$$\boldsymbol{x_{critcal} = \pm\frac{a}{\sqrt{2}}}$$

Each root gives a force with magnitude of $\frac{4GMm}{3\sqrt{3}a^2}$ pointing towards the origin.

**Solution continues on the next page…**

d) To use $(1+\delta)^n \approx 1 + n\delta$ we first need to set up the equation with a suitable $\delta \ll 1$.
We are told $x \ll a$. Dividing both sides by $a$ gives $\frac{x}{a} \ll 1$. Now rearrange the denominator of part b such that $\frac{x}{a}$ appears. To do this, factor out $(a^2)^{3/2}$. It should look something like this where the binominal expansion was used in the final step.

$$\vec{F} = \frac{-2GMmx}{(a^2+x^2)^{3/2}}\hat{\imath}$$

$$\vec{F} = \frac{-2GMmx}{\left(a^2 + a^2\frac{x^2}{a^2}\right)^{3/2}}\hat{\imath}$$

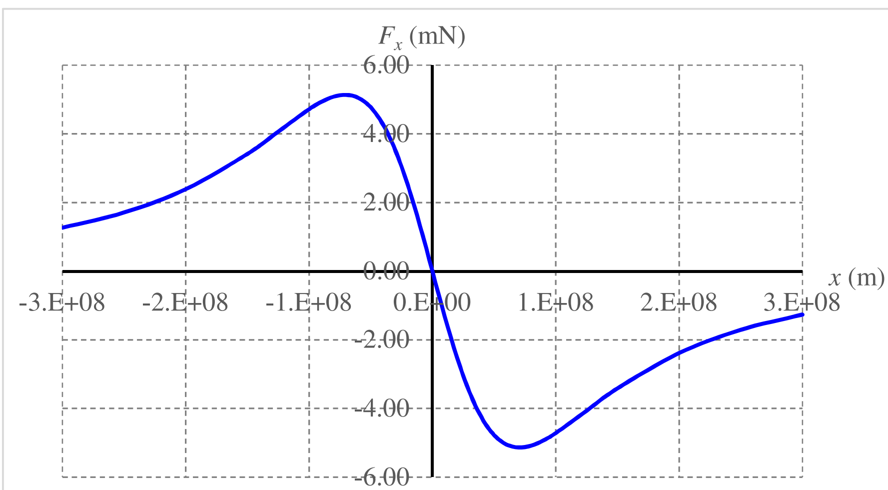
$$\vec{F} = \frac{-2GMmx}{(a^2)^{3/2}\left(1 + \frac{x^2}{a^2}\right)^{3/2}}\hat{\imath}$$

$$\vec{F} = \frac{-2GMmx}{a^3\left(1 + \frac{x^2}{a^2}\right)^{3/2}}\hat{\imath}$$

$$\vec{F} = -\frac{2GMmx}{a^3}\left(1 + \frac{x^2}{a^2}\right)^{-3/2}\hat{\imath}$$

$$\vec{F} \approx -\frac{2GMmx}{a^3}\left(1 - \frac{3x^2}{2a^2}\right)\hat{\imath}$$

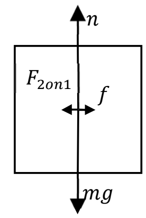
e) I used units of milliNewtons for force to clean it up a bit.
I forgot to choose a decent prefix for $x$…my bad. Probably should've used km for familiarity.
Notice the graph is linear with a negative slope close to the origin.
Any time you have a linear force graph with a negative slope you may treat the force like a spring!



**13.15**

a) $s = 17.3 \text{ cm}$

b) The force attracting the blocks to each other has magnitude $F_{2on1} = \frac{Gm^2}{(s+0.10 \text{ m})^2} = 8.9 \text{ μN}$.




Since the blocks are near earth we may use $mg$ for the force towards the earth as seen in the FBD. If the block is on the verge of slipping we may use $f = \mu_s n = \mu_s mg$. Solving for $\mu_s$ gives

$$\mu_s = \frac{F_{2on1}}{mg} = 9.1 \times 10^{-9}$$

c) This coefficient is absurdly small. I can't imagine any way the blocks could ever slide towards each other. People sized objects cannot use gravity to attract one another. Except in extremely sensitive apparatus, we may completely ignore gravitation forces between two objects near the earth's surface.

**13.16** The FBD and force equation should lead to $M = \frac{4\pi^2 R^3}{\mathbb{T}^2 G} \approx 2.0 \times 10^{30}$ kg.

Notice the mass of the orbiting body drops out while the central mass remains. Perhaps you were curious how astronomers determined the masses of all the planets and the sun. If one observes a moon or satellite and determine the orbital period and radius. From this one can determine the mass of the planet at the center of the orbit! Notice, however, this requires knowing a value of $G$. If you haven't already, this is perhaps an interesting time to read about the Cavendish (1797) and Schiehallion (1774) experiments online. These experiments determined the density of the earth (which in turn could determine the value of $G$).

**13.17** Approximately 35900 km. I will assume the planet is mass $M$ and radius $R$. The satellite is mass $m$ and moves in a circular orbit with speed $v$ at altitude $h$. Notice the radius of the circular orbit is $r = R + h$.

$$\frac{GmM}{r^2} = \frac{mv^2}{r}$$

$$\frac{GM}{r} = v^2$$

$$\frac{GM}{r} = \left(\frac{2\pi r}{\mathbb{T}}\right)^2$$

$$\frac{GM}{r} = \frac{4\pi^2 r^2}{\mathbb{T}^2}$$

$$r^3 = \frac{GM\mathbb{T}^2}{4\pi^2}$$

$$r = \left(\frac{GM\mathbb{T}^2}{4\pi^2}\right)^{1/3}$$

$$R + h = \left(\frac{GM\mathbb{T}^2}{4\pi^2}\right)^{1/3}$$

$$\boldsymbol{h = \left(\frac{GM\mathbb{T}^2}{4\pi^2}\right)^{1/3} - R}$$

BUT WAIT! We know the period of the satellite is supposed to be the same as the time for earth to rotate once. This means the period of the orbit is 1 day = 24.0 hrs = 86400 s. Leaving off units to reduce clutter:

$$h = \left(\frac{(6.67 \times 10^{-11})(5.97 \times 10^{24})(8.964 \times 10^4)^2}{4\pi^2}\right)^{1/3} - 6.37 \times 10^6 \approx 3.59 \times 10^7 \text{ m} = 35900 \text{ km}$$

**13.18**

a) Similar to problem **13.8** we find

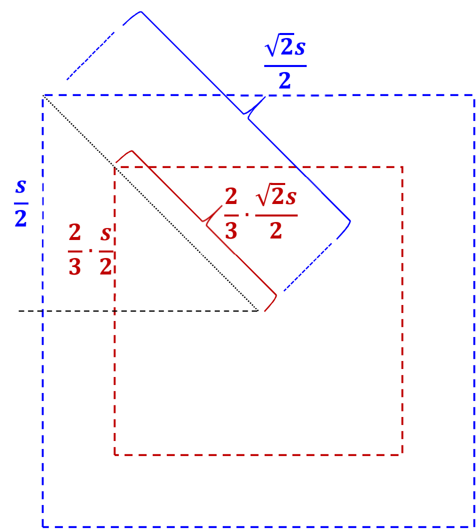
$$U_i = U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34} = -4\frac{Gm^2}{s} - 2\frac{Gm^2}{\sqrt{2}s} = -5.414\frac{Gm^2}{s}$$

After traveling $\frac{1}{3}$of the distance to the center the square will be scaled by a factor of $\frac{2}{3}$. This implies the distances are lowered by $\frac{2}{3}$ which multiplies the potential energy by $\frac{3}{2}$ (inversely proportional to *s*). Lastly, since all four are moving the kinetic energy is $K = 4\left(\frac{1}{2}mv^2\right)$.

$$v = \sqrt{\frac{2(U_i - U_f)}{4m}} = \sqrt{\frac{\left(U_i - \frac{3}{2}U_i\right)}{2m}} = \sqrt{-\frac{U_i}{4m}} = 1.16\sqrt{\frac{Gm}{s}}$$

b) If you go 1/3 of the way to the center, you still have 2/3 left to go.
The distances all scale by $\frac{2}{3}$.
Force is inversely proportional to distance squared.
The forces increase by a factor of $\frac{9}{4} = 2.25$.

If the previous sentences don't make sense, let $r' = \frac{2}{3}r$ and $F'$ be the new force mag at $r'$. See figure at right.




$$F' = \frac{Gmm}{r'^2}$$

$$F' = \frac{Gm^2}{\left(\frac{2}{3}r\right)^2}$$

$$F' = \frac{Gm^2}{\left(\frac{2}{3}\right)^2 r^2}$$

$$F' = \frac{Gm^2}{\frac{4}{9}r^2}$$

$$F' = \frac{9}{4}\cdot\frac{Gm^2}{r^2}$$

Now one can take the ratio

$$ratio = \frac{F'}{F} = \frac{\frac{9}{4}\cdot\frac{Gm^2}{r^2}}{\frac{Gm^2}{r^2}} = \frac{9}{4}$$

c) As the particles move towards the center the force magnitudes (and thus acceleration magnitudes) increase at an ever increasing rate. Both the acceleration magnitude and jerk magnitude increase as the object moves towards the center. As the object accelerates it covers distances close to the center more rapidly than near the edges which further increases jerk as the objects get closer to impact. Said another way

$$a = \frac{F}{m} = k\frac{1}{r^2}$$

where $k$ is some constant from the numbers $G$ and $m$.

$$|J| = \left|\frac{da}{dt}\right| = \left|\frac{\partial a}{\partial r}\frac{dr}{dt}\right| = \left|\frac{\partial a}{\partial r}v\right| = 2k\frac{1}{r^3}v$$

We see the magnitude of jerk increases with speed and decreasing radius as expected.

**13.19 Going left to right in the figure the masses should be $2M$, $m$, and $M$.**

WATCH OUT! This seemingly innocuous question has three cases to consider for all parts:

- Little $m$ to the *left* of masses $2M$ & $M$ $(x < -d)$
- Little $m$ *between* masses $2M$ & $M$ $(-d < x < d)$
- Little $m$ to the *right* of masses $2M$ & $M$ $(x > d)$

a) When you look at the denominators we see the binomial terms $d - x$ and $d + x$ appearing. The only parameter free to vary is $x$ so $x$ is the thing that can be really small. The other distance in the problem is $d$. These things all indicate we should consider what happens when $\delta = \frac{x}{d} \ll 1$.

b) Notice first that when $x$ is small there is no longer need for the absolute value. $U$ gives

$$U = -GM\left(\frac{2m}{d\left(1+\frac{x}{d}\right)} + \frac{m}{d\left(1-\frac{x}{d}\right)} + M\right)$$

$$U = -\frac{GM}{d}\left(\frac{2m}{1+\frac{x}{d}} + \frac{m}{1-\frac{x}{d}} + M\right)$$

$$U = -\frac{GM}{d}\left(2m\left(1+\frac{x}{d}\right)^{-1} + m\left(1-\frac{x}{d}\right)^{-1} + M\right)$$

$$U \approx -\frac{GM}{d}\left(2m\left(1-\frac{x}{d}\right) + m\left(1+\frac{x}{d}\right) + M\right)$$

$$U \approx -\frac{GM}{d}\left(M + 3m - m\frac{x}{d}\right) = \frac{GMm}{d^2}x - \frac{GM}{d}(M + 3m) = (slope)x + intercept$$

Notice everything is constant except $x$. The function is now linear with $slope = \frac{GMm}{d^2}$. Look at the plot of $U$ vs $x$ in the solutions to **13.6** and notice the negative intercept and, for $\frac{x}{d} \ll 1$, the positive slope.

Performing the expansion on the force gives

$$F_x = \frac{GMm}{d^2}\left[\left(1-\frac{x}{d}\right)^{-2} - 2\left(1+\frac{x}{d}\right)^{-2}\right]$$

$$F_x \approx \frac{GMm}{d^2}\left[\left(1+2\frac{x}{d}\right) - 2\left(1-2\frac{x}{d}\right)\right]$$

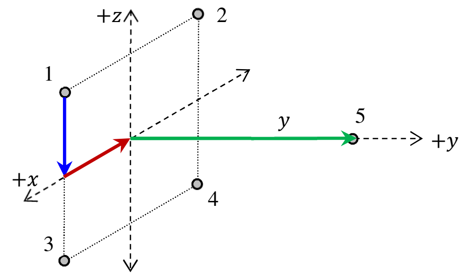
$$F_x \approx 6\frac{GMm}{d^3}x - \frac{GMm}{d^2}$$

Check the plot of $F_x$ vs $x$ with the solution to problem **13.6**. Notice the negative intercept and, for $\frac{x}{d} \ll 1$, the positive slope. Note: checking all this crap helped me catch mistakes in my original solution to **13.6**.

c) If you are FAR from the origin $(x \gg R)$, one can simply consider $M$ & $2M$ as being a single mass $3M$ located at the origin to get a quick *estimate* of the *force*. If you are concerned with *energy*, the small mass is so far away the energy contributions associated with it are approximately negligible!

**13.20**

a) In 3D using unit vectors is totally the way to go IMHO. Going from 1 to 5 gives the displacement vector, distance, and unit vector shown below.




$$\vec{r}_{1to5} = -\frac{s}{2}\hat{\imath} + y\hat{\jmath} - \frac{s}{2}\hat{k}$$

$$r_{1to5} = \sqrt{\left(\frac{s}{2}\right)^2 + y^2 + \left(\frac{s}{2}\right)^2} = \left(\frac{s^2}{2} + y^2\right)^{1/2}$$

$$\hat{r}_{1to5} = \frac{-\frac{s}{2}\hat{\imath} + y\hat{\jmath} - \frac{s}{2}\hat{k}}{\left(\frac{s^2}{2} + y^2\right)^{1/2}}$$

The force is given by

$$\vec{F}_{\mathbf{1on5}} = \frac{Gm_1m_5}{r_{1to5}^2}\left(-\hat{r}_{1to5}\right)$$

$$\vec{F}_{\mathbf{1on5}} = \frac{Gm^2}{\left[\left(\frac{s^2}{2} + y^2\right)^{1/2}\right]^2}\left(-\frac{-\frac{s}{2}\hat{\imath} + y\hat{\jmath} - \frac{s}{2}\hat{k}}{\left(\frac{s^2}{2} + y^2\right)^{1/2}}\right)$$

$$\vec{F}_{\mathbf{1on5}} = \frac{Gm^2}{\left(\frac{s^2}{2} + y^2\right)^{3/2}}\left(\frac{s}{2}\hat{\imath} - y\hat{\jmath} + \frac{s}{2}\hat{k}\right)$$

Before moving on check the direction of the force 1 exerts <u>on</u> 5.
It points up $(+\hat{k})$, out of the page $(+\hat{\imath})$, and towards the center of the square $(-\hat{\jmath})$.
This is the correct general direction the force should point if 1 is attracting 5.

b) By symmetry, we expect the $\hat{\imath}$ & $\hat{k}$ parts to drop out. The final force should be in the $-\hat{\jmath}$ direction.

c) Since there are force mass all contributing the same $-\hat{\jmath}$ term the *NET* gravitational force is

$$\vec{F}_{NET\mathbf{on5}} = \frac{-4Gm^2y}{\left(\frac{s^2}{2} + y^2\right)^{3/2}}\hat{\jmath}$$

Note: if $y \ll s$ the denominator simplifies without the need of the binomial expansion giving

$$\vec{F}_{NET\mathbf{on5}} \approx \frac{-8\sqrt{2}Gm^2y}{s^3}\hat{\jmath} \qquad \text{for } y \ll s$$

**Notice it is *negative* and *linear*.** It may be treated like a spring with effective spring constant

$$k_{eff} = \frac{-8\sqrt{2}Gm^2}{s^3}$$

$$\vec{F}_{NET\mathbf{on5}} \approx -k_{eff}y\hat{\jmath} \qquad \text{for } y \ll s$$

Note: if $y \gg s$ the denominator simplifies without the need of the binomial expansion giving

$$\vec{F}_{NET\mathbf{on5}} \approx \frac{-4Gm^2}{y^2}\hat{\jmath} \qquad \text{for } y \gg s$$

This seems reasonable. If $y \gg s$ the square would seem incredibly small compared to $y$. The four masses would eseentially all be right on top of each other at the origin. We could then think of the four masses as a single point mass $4m$ and get the same result as the expansion. This is a cool check on your work!

**13.21**

a) To get potential energy we want to use

$$U_{total} = sum\ of\ U's for\ each\ pair\ of\ masses$$

$$U_{total} = U_{12} + U_{13} + U_{23}$$

$$U_{total} = \left(-\frac{Gm_1m_2}{distance_{12}}\right) + \left(-\frac{Gm_1m_3}{distance_{13}}\right) + \left(-\frac{Gm_2m_3}{distance_{23}}\right)$$

$$U_{total} = \left(-\frac{Gm_1m_2}{r_{12}}\right) + \left(-\frac{Gm_1m_3}{r_{13}}\right) + \left(-\frac{Gm_2m_3}{r_{23}}\right)$$

Note: it seems easiest to let $m_1 = m$, $m_2 = 2m$, and $m_3 = 3m$.

**The distances between the masses are $\boldsymbol{r_{12} = 10d, r_{13} = 13d}$, & $\boldsymbol{r_{23} = 22.20d}$.** Therefore

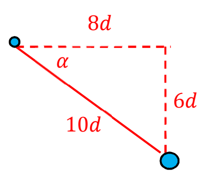
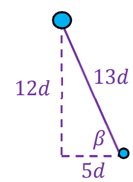
$$U_{total} = \left(-\frac{G(m)(2m)}{10d}\right) + \left(-\frac{G(m)(3m)}{13d}\right) + \left(-\frac{G(2m)(3m)}{22.20d}\right)$$

Notice you can factor out $-\frac{Gm^2}{d}$ giving

$$U_{total} = -\frac{Gm^2}{d}\left[\frac{2}{10} + \frac{3}{13} + \frac{6}{22.20}\right] = -0.701\frac{Gm^2}{d}$$

**Solution continues on the next page.**

b) To get force we can either get the magnitude then split into components (using SOH CAH TOA) OR do the whole unit vector thing. The previous problem had 3D forces and the unit vector method was the way to go. See that problem if you need practice with the unit vector method. Since this problem is planar (all masses in $xy$-plane) we may as well do the magnitude and SOH CAH TOA thing…

| **Force of 2$m$ on $m$** | **Force of 3$m$ on $m$** |
|---|---|
| $F_{2on1} = magnitude\ of\ force$ <br> $F_{21} = \dfrac{Gm_1m_2}{(distance_{21})^2}$ <br> Notice: no minus sign because this is *magnitude*. <br> $F_{21} = \dfrac{Gm_1m_2}{r_{21}^2}$ <br> $F_{21} = \dfrac{G(m)(2m)}{(10d)^2}$ <br> $F_{21} = \dfrac{Gm^2}{50d^2}$ <br> $\vec{F}_{21}$ points *down* and to the *right*. <br>   <br> $\sin\alpha = \dfrac{6}{10} = 0.6000$ <br> $\cos\alpha = \dfrac{8}{10} = 0.8000$ <br> $\vec{F}_{21} = \dfrac{Gm^2}{50d^2}\big((+\cos\alpha\,\hat{\imath}) + (-\sin\alpha\,\hat{\jmath})\big)$ <br> $\vec{F}_{21} = \dfrac{Gm^2}{50d^2}\big((+0.8000\hat{\imath}) + (-0.6000\hat{\jmath})\big)$ <br> $\vec{F}_{2on1} = \dfrac{Gm^2}{d^2}\big((+0.01600\hat{\imath}) + (-0.01200\hat{\jmath})\big)$ | $F_{3on1} = magnitude\ of\ force$ <br> $F_{31} = \dfrac{Gm_1m_3}{(distance_{31})^2}$ <br> Notice: no minus sign because this is *magnitude*. <br> $F_{31} = \dfrac{Gm_1m_3}{r_{31}^2}$ <br> $F_{31} = \dfrac{G(m)(3m)}{(13d)^2}$ <br> $F_{31} = \dfrac{3Gm^2}{169d^2}$ <br> $\vec{F}_{31}$ points *up* and to the *left*. <br>   <br> $\sin\beta = \dfrac{12}{13} = 0.9231$ <br> $\cos\beta = \dfrac{5}{13} = 0.3846$ <br> $\vec{F}_{31} = \dfrac{3Gm^2}{169d^2}\big((-\cos\beta\,\hat{\imath}) + (+\sin\beta\,\hat{\jmath})\big)$ <br> $\vec{F}_{31} = \dfrac{3Gm^2}{169d^2}\big((-0.3846\hat{\imath}) + (+0.9231\hat{\jmath})\big)$ <br> $\vec{F}_{3on1} = \dfrac{Gm^2}{d^2}\big((-0.006827\hat{\imath}) + (+0.01639\hat{\jmath})\big)$ |

$$\vec{F}_{\mathbf{NET}on1} = \vec{F}_{2on1} + \vec{F}_{3on1}$$

$$\vec{F}_{\mathbf{NET}on1} = \frac{Gm^2}{d^2}\big((+0.01600\hat{\imath}) + (-0.01200\hat{\jmath})\big) + \frac{Gm^2}{d^2}\big((-0.006827\hat{\imath}) + (+0.01639\hat{\jmath})\big)$$

$$\vec{F}_{\mathbf{NET}on1} = \frac{Gm^2}{d^2}\left((+0.01600 - 0.006827)\hat{\imath} + (-0.01200 + 0.01639)\hat{\jmath}\right)$$

$$\vec{F}_{\mathbf{NET}on1} = \frac{Gm^2}{d^2}\left((0.00917)\hat{\imath} + (0.00439)\hat{\jmath}\right)$$

$$\boldsymbol{F_{\mathrm{NET}on1} = 0.0102\frac{Gm^2}{d^2}}$$

c) From the sketch at right you can see

$$\phi = \tan^{-1}\left(\frac{0.00439}{0.00917}\right) = 25.6°$$