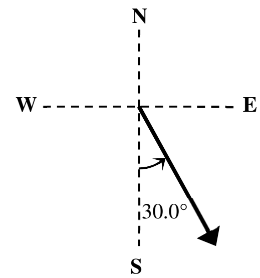
Vectors, Vector Components, Scalar Components

A vector has magnitude and direction. An example you have already seen is velocity which has magnitude (speed) and direction (heading). Suppose we have a velocity that is 15.0 m/s with heading 30.0° east of south. This vector is shown in the figure at right. The cardinal directions (north, south, east, & west) are shown as they are typically drawn with **N** as the positive y-axis and **E** as the positive x-axis. The standard conventions for describing the vector, vector components, and scalar components are shown below. Please pay particular attention to when the $\hat{\imath}$ and $\hat{\jmath}$ are included. Also notice, "east of south" implies you first align with south and then go east from it.



Vector (Polar Form 1)	$\vec{v} = 15.0\,\frac{\mathrm{m}}{\mathrm{s}}$ @ 30.0° E of S
Vector (Polar Form 2)	$\vec{v} = 15.0\,\frac{\mathrm{m}}{\mathrm{s}}$ @ 300.0°
Vector (Polar Form 3)	$\vec{v} = 15.0\,\frac{\mathrm{m}}{\mathrm{s}}$ @ -60.0°
Vector (Cartesian Form)	$\vec{v} = 7.50\,\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} - 13.0\,\frac{\mathrm{m}}{\mathrm{s}}\hat{\jmath}$
Vector Components	$\overrightarrow{v_x} = 7.50\,\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$ & $\overrightarrow{v_y} = -13.0\,\frac{\mathrm{m}}{\mathrm{s}}\hat{\jmath}$
Scalar Components	$v_x = 7.50\,\frac{\mathrm{m}}{\mathrm{s}}$ & $v_y = -13.0\,\frac{\mathrm{m}}{\mathrm{s}}$

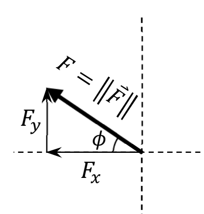
The following are several points worth stressing:

- In polar form 1, my convention is often to reference the closest cardinal direction.
- In polar forms 2 & 3, by convention angles are referenced to the positive *x*-axis.
- Vector and scalar components both have ± signs
- Include the arrow ($\overrightarrow{\quad}$) the $\hat{\imath}$ and $\hat{\jmath}$ on the *vector* components but not on the *scalar* components
- Do not say "-60° below" the *x*-axis. Just as "-2 to the left" implies "+2 to the right", "-60° *below* the *x*-axis" implies **"**+60° *above* the *x*-axis".

Some other useful vector information

The symbol indicating the *magnitude* of a vector is written in one of two ways:

$$\text{the } magnitude \text{ of vector } \vec{F} = \begin{cases} \|\vec{F}\| \\ \textbf{or} \\ F \end{cases}$$



One computes the magnitude using the following equation:

$$F = \sqrt{F_x^2 + F_y^2}$$

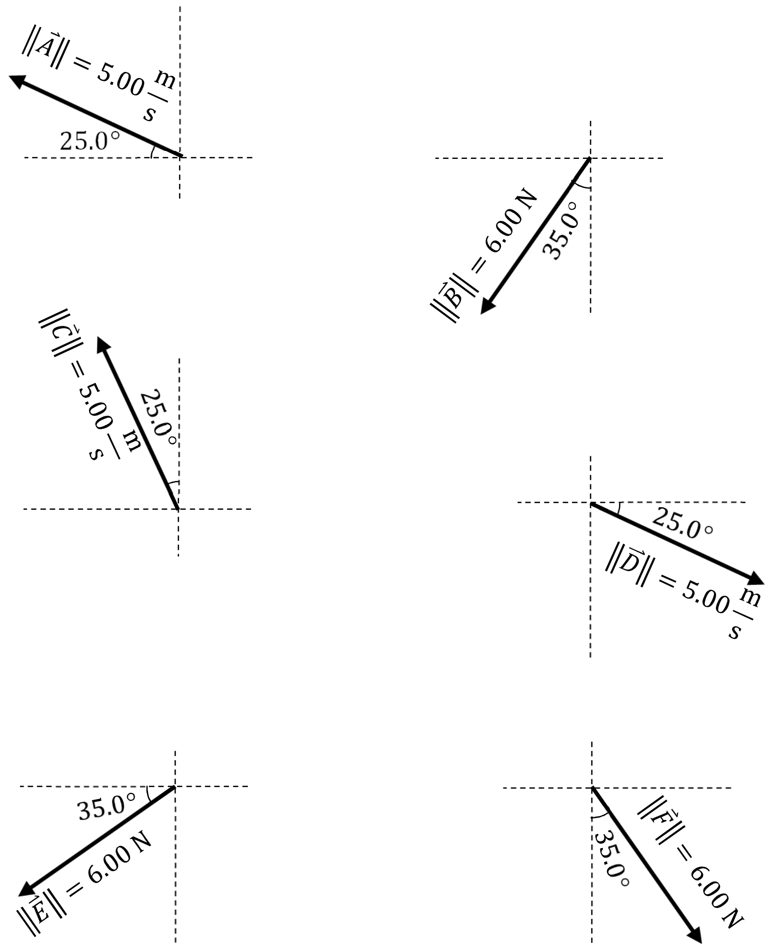
One can determine the direction using SOH CAH TOA (see figure at right).

$$\phi = \tan^{-1}\left|\frac{F_y}{F_x}\right|$$

Note: I use the absolute value in the above equation but also draw pictures to ensure I understand the direction.
I feel doing this process forces me to check & double check my own understanding.
I let the arrowheads in my picture remind me which terms are positive or negative.
This may seem strange now, but pretty much all physicists do this all the time in chapters 5 & 6 (forces).

Note: for 3D vectors the magnitude equation becomes $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$.

3.1 Consider the following vectors. Write down each vector in Cartesian form.



3.2 Determine the magnitude and direction of each vector. To clearly label the direction, sketch a picture and label the one of the angles.

$\vec{G} = (-12.00\hat{\imath} - 5.00\hat{\jmath})\text{m}$	$\vec{H} = (-4.00\hat{\imath} + 3.00\hat{\jmath})\text{m}$

3.3 Switching from magnitude and direction (polar form) to unit-vector notation (Cartesian form).
It is easy to mix up ± signs or switch sine and cosine when determining components. It is worth it to verify you are doing this correctly before moving on to performing mathematical operations with vectors. Write each vector using unit-vector notation. Some answers turn out the same. After computing components, check the signs and sizes of each component with a sketch.

$\vec{A} = 30.0\,\frac{\text{m}}{\text{s}}$ directed $15\underline{0}°$ from the positive x-axis	$\vec{D} = 30.0\,\frac{\text{m}}{\text{s}}$ directed 30.0° below the negative x-axis
$\vec{B} = 30.0\,\frac{\text{m}}{\text{s}}$ directed -30.0° from the positive x-axis	$\vec{E} = 30.0\,\frac{\text{m}}{\text{s}}$ directed 30.0° west of north
$\vec{C} = 30.0\,\frac{\text{m}}{\text{s}}$ directed 30.0° above the negative x-axis	$\vec{F} = 30.0\,\frac{\text{m}}{\text{s}}$ directed 30.0° north of west

Note to teachers: in most vector addition solutions I used green for $\vec{A}$, blue for $\vec{B}$, red for $\vec{C}$, and black for $\vec{R}$.

Component-wise Vector Addition
When adding vectors component-wise (using $\hat{\imath}$ and $\hat{\jmath}$), use the following procedure
Example is worked on next two pages (shown in two different styles).

1) Split the vectors into components using trig and/or geometry.
2) Double check the signs and relative sizes
    a. Did you flip a sine with a cosine by accident?
    b. Did you forget to put in a minus sign?
3) Add the components together to obtain the resultant vector
4) Convert the components to polar form (magnitude and direction)

Note: the order in which you add vectors is unimportant.

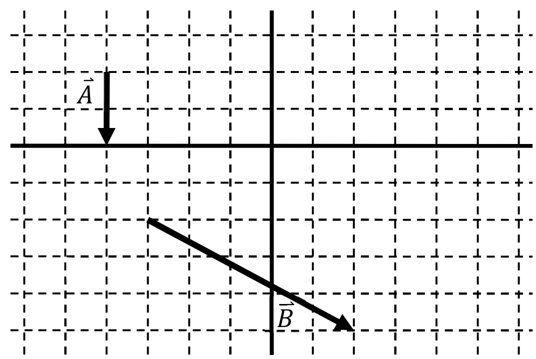
Graphical Vector Addition
To add vectors graphically, follow this procedure.

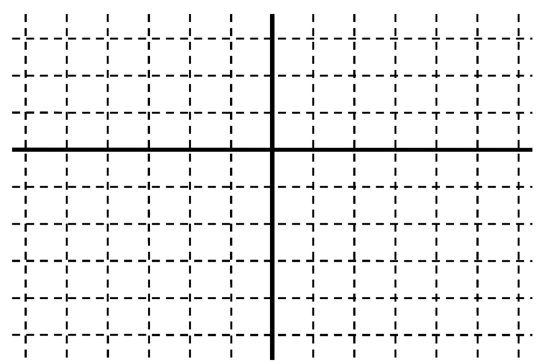
1) Place the tail of the first vector at the origin.
2) At the tip of the first vector, sketch a tiny new coordinate system. The positive *y*-direction still points straight up.
3) Put the tail of the second vector at the origin of this new coordinate system.
4) Repeat until all vectors are used.
5) The resultant is found by connecting the first tail to the last tip.

3.4 Assume the spacing between adjacent gridlines is 10.0 $\frac{\mathrm{m}}{\mathrm{s}}$.

Here the vectors $\vec{A}$ and $\vec{B}$ represent velocity vectors with a magnitude (speed) and direction (heading). Manipulation of these vectors would be important to perhaps an airplane pilot. One might be wind velocity. Another might be the plane velocity.

a) Write down each vector in Cartesian component form.
b) Write down each vector in polar form 1 (angle to the nearest cardinal direction).
c) A third vector $\vec{C}$ has speed 50.0 $\frac{\mathrm{m}}{\mathrm{s}}$ with heading 36.87° west of north. Determine this vector in Cartesian component form and draw it somewhere on the upper grid. Recall the tail need not be at the origin.
d) Find a partner. Have each person choose a different order for the vectors and do graphical addition method (tail-to-tip) to add your three vectors on the fresh grid at right. Verify you get the same resultant.
e) Verify you obtain the same result using component wise addition.





Note: for this special case graphical addition was probably easier. In general component-wise addition is faster and more precise.

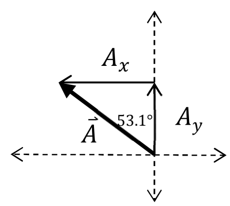
Example of vector addition

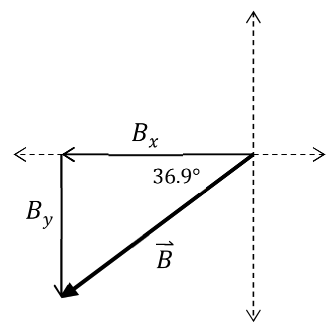
Two vectors are given as $\vec{A} = 5.00 \;@\; 53.1°$ W of N and $\vec{B} = 10.00 \;@\; 36.9°$ S of W.
Determine $\vec{R} = \vec{A} + \vec{B}$.

Procedure

1) Sketch each vector to get a feel for where they point and label the angles.
2) Rewrite in Cartesian form.
    a. Double check the signs of each component of each vector.
    b. Based on your sketch, verify you appropriately used sine/cosine for each component.
3) Add $\hat{\imath}$'s to $\hat{\imath}$'s and $\hat{\jmath}$'s to $\hat{\jmath}$'s.
4) Sketch the final vector $\vec{R}$.
5) Use your sketch to determine $\vec{R}$ in polar form.
    a. Use the Pythagorean theorem to determine the magnitude.
    b. Use SOH CAH TOA to get the angle (usually $\tan^{-1}$).
6) Verify the angle is correct using graphical addition (tail-to-tip method).

$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$
where $A_x = -5.00 \sin 53.1°$ and $A_y = +5.00 \cos 53.1°$.
The ±'s come from the arrowheads in sketch. Notice that cosine and sine are switched from the standard math convention! In standard math notation the angle is always to the horizontal.





$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$
where $B_x = -10.0 \cos 36.9°$ and $B_y = -10.0 \sin 36.9°$.
The ±'s come from the arrowheads in sketch.

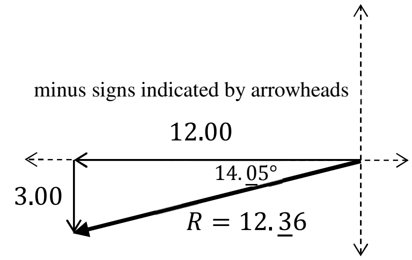
Now we see

$$\begin{array}{rl} \vec{A} = & -5.00 \sin 53.1°\,\hat{\imath} + \;\; 5.00 \cos 53.1°\,\hat{\jmath} \\ \underline{+\vec{B} =} & \underline{-10.0 \cos 36.9°\,\hat{\imath} - 10.0 \sin 36.9°\,\hat{\jmath}} \\ \vec{R} = & \quad -11.9\underline{95}\,\hat{\imath} \quad + \quad -3.0\underline{0}2\hat{\jmath} \end{array}$$

Notice the extra sig fig on the $\hat{\imath}$ term coming from the addition. Now sketch the final vector.
We see

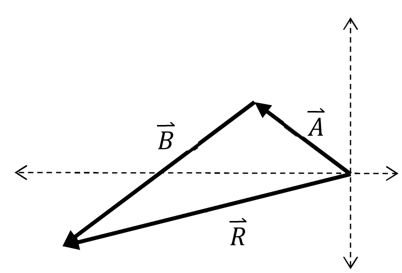
$$R = \|\vec{R}\| = \sqrt{(-11.9\underline{95})^2 + (-3.0\underline{0}2)^2} = 12.\underline{36}$$

$$\theta = \tan^{-1}\left(\frac{-3.0\underline{0}2}{-11.9\underline{95}}\right) = 14.\underline{05}°$$



WATCH OUT! Calculators output a number *in either quadrant I or IV*. By looking at my picture I can tell this is the same thing as $14.1°$ below the *negative* x-axis. Another way to correctly describe it is $14.1° + 180.0° = 194.1°$ from the *positive* x-axis. To avoid this confusion, I usually ignore all minus signs when determining the angle and use my sketch of the vector $\vec{R}$.

Lastly, I did the graphical vector addition just to verify it looks about right. I see my answer from above matches up fairly well as it is pointing at approximately the correct angle in the correct quadrant.



TIP: ALWAYS DO A SKETCH TO GET THE QUADRANT CORRECT.

Same example problem as previous page done with a different style

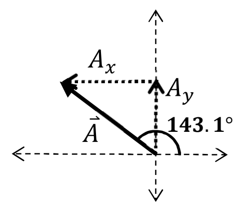
Two vectors are given as $\vec{A} = 5.00$ @ $53.1°$ W of N and $\vec{B} = 10.0$ @ $36.9°$ S of W.
Determine $\vec{R} = \vec{A} + \vec{B}$.

The angle from the positive x-axis to the vector $\vec{A}$ is $\alpha = 90.0° + 53.1° = 143.1°$.
The angle from the positive x-axis to the vector $\vec{B}$ is $\beta = 180.0° + 36.9° = 216.9°$.
Alternatively, many resources state the angle from the positive x-axis to the vector $\vec{B}$ is $\beta = -143.1°$.

Strategy if all angles are defined to positive x-axis:

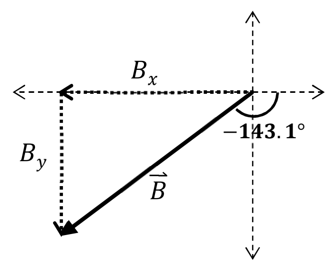
1) The x-component of the vector ALWAYS USES COSINE.
2) The y-component of the vector ALWAYS USES SINE.
3) The sine & cosine functions will automatically determine the sign of the vectors for you.
4) DO NOT manually adjust sings of components after the fact based on the direction of the arrowheads.
5) DO verify the signs implied by the arrowheads match the signs in your computation.

$$\begin{aligned}
\vec{A} &= A_x\hat{\imath} + A_y\hat{\jmath} \\
\vec{A} &= A\cos\alpha\,\hat{\imath} + A\sin\alpha\,\hat{\jmath} \\
\vec{A} &= 5.00\cos 143.1°\,\hat{\imath} + 5.00\sin 143.1°\,\hat{\jmath} \\
\vec{A} &= -3.9\underline{9}8\hat{\imath} + 3.0\underline{0}2\hat{\jmath}
\end{aligned}$$



Note: correct ± signs came from computing the sine & cosine functions in your calculator.

$$\begin{aligned}
\vec{B} &= B_x\hat{\imath} + B_y\hat{\jmath} \\
\vec{B} &= B\cos\beta\,\hat{\imath} + B\sin\beta\,\hat{\jmath} \\
\vec{B} &= 10.0\cos(-143.1°)\,\hat{\imath} + 10.0\sin(-143.1°)\,\hat{\jmath} \\
\vec{B} &= -7.9\underline{9}7\hat{\imath} + -6.0\underline{0}4\hat{\jmath}
\end{aligned}$$



Note: correct ± signs came from computing the sine & cosine functions in your calculator.

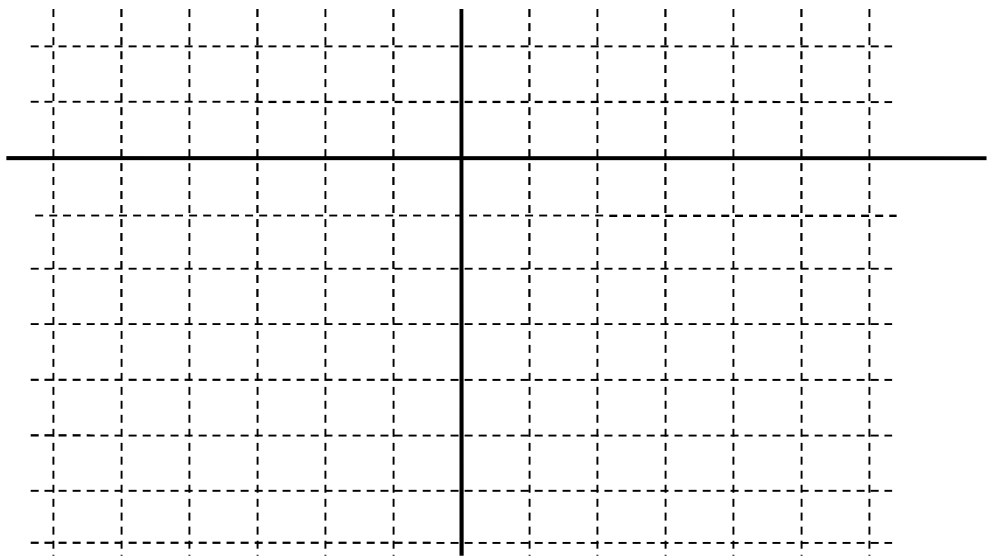
Now we see

$$\begin{aligned}
\vec{A} &= -3.9\underline{9}8\hat{\imath} + 3.0\underline{0}2\hat{\jmath} \\
+\vec{B} &= -7.9\underline{9}7\hat{\imath} - 6.0\underline{0}4\hat{\jmath} \\
\hline
\vec{R} &= -11.9\underline{9}5\,\hat{\imath} - 3.0\underline{0}2\hat{\jmath}
\end{aligned}$$

Which method is best?

I argue the first method is best when doing problems for some problems (most in chapter 5, 6, 12, etc).
At the same time, this 2nd method is preferable if you plan to do any coding with vectors.
Both are styles useful.
Learn both styles.

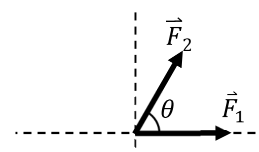
3.5 Bob first walks 20.0 m at 60.0° S of E, then 30.0 m at 30.0° S of W, then an unknown direction and distance. Bob's final position is 5.00 m due east of his initial position. Use component wise addition to determine the third leg of the trip (distance and direction). Check your work by sketching the graphical addition on the grid below. Note: In this example, things won't line up perfectly on the gird lines. Let each tick mark below indicate 5 m.



3.6 Add the following vectors: $\vec{A} = 8.00$ @ 30.0° N of W, $\vec{B} = 10.0$ @ 40.0° W of S. Express your result in both Cartesian and polar forms. Include a sketch showing the graphical vector addition.

3.7 You are told $\vec{R} = \vec{A} + \vec{B} + \vec{C}$. $\vec{A} = 8.00\ due\ north$, $\vec{B} = 6.00$ @ 30.0° E of S, and $\vec{R} = 10.0\ due\ west$. Determine the unknown vector $\vec{C}$. The usual Cartesian, polar, and graphical answers are expected.

3.7½ For this problem be very careful to distinguish between force *vectors* and force *magnitudes*. Two force vectors ($\vec{F}_1$ & $\vec{F}_2$) have identical magnitude F. The first force is aligned with the positive x-axis. The second can be applied at any angle θ between 0° & 180°.

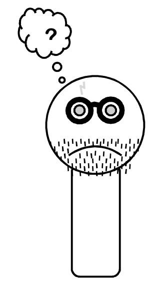


a) At what angle should $\vec{F}_2$ be placed to cause a net force magnitude of $1.75F$?
   Hint: first pretend you actually know the angle and determine $\vec{F}_{NET} = \vec{F}_1 + \vec{F}_2$ as usual. Next determine the magnitude as usual. At this point you have a relationship between the net force magnitude (given in problem statement) and the angle (what to find).
b) What direction does the net force vector point when $F_{NET} = 1.75F$?
c) At what angle should $\vec{F}_2$ be placed to cause a net force magnitude of $\frac{F}{2}$?
d) At what angle should $\vec{F}_2$ be placed to cause a net force magnitude of F?
e) What angle gives no net force?
f) What is the maximum possible net force magnitude? What angle should be used to cause maximum net force?

3.8 The Story of Mr. Boanventur…

A materials engineering student named Mr. Boanventur went out into the middle of a large flat field on a very foggy day. He proceeded to smoke a strain of medical marijuana known as 3SL. This strain was peculiar. Anyone who smoked it began to walk three straight line displacements in a manner not at all unlike an spooky movie plot. Mr. Boanventur smoked a lot of it.

At first Boanventur wasn't too messed up. For the first leg of the trip he dutifully noted he walked 100.0 m heading 36.9° N of E. Then the drug really kicked in. For the next leg of his trip he was careful to record the distance traveled as 80.0 m but totally forgot to measure his heading. On the last leg, Boanventur felt so bad about forgetting the angle that's all he could think about. He recorded the heading of his last displacement as 20.0° W of S. Of course, this time, wouldn't you know it, he forgot to record the distance traveled. Note: at some point he fell and scratched his face in a manner that made him look like Harry Potter with a beard.



Miraculously, Boanventur somehow made it back to the starting position despite the thick fog. He found a bag of crisps and began to cry tears of joy. He proceeded to apply to a prestigious local university and accidentally misspelled his name on his transfer application…for real.

Determine the heading of the second leg of the journey as well as the length of the second part of the journey. Is it impossible to determine? The vector equation gives us two equations (one eqt'n each for x- & y-directions) and we only have two unknowns…

DOT PRODUCTS

The equations and useful facts for dot products are given in the table below.

$$\vec{A}\cdot\vec{B} = \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right)\cdot\left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right) = A_xB_x + A_yB_y + A_zB_z \qquad 3.1$$

$$\vec{A}\cdot\vec{B} = AB\cos\theta_{AB} \qquad 3.2$$

$$\theta_{AB} = \text{the angle between } \vec{A}\ \&\ \vec{B} \qquad 3.3$$

$$\text{The magnitude of a vector is } A = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad 3.4$$

$$\hat{\imath}\cdot\hat{\imath} = 1 \qquad \hat{\imath}\cdot\hat{\jmath} = 0 \qquad \hat{\jmath}\cdot\hat{\imath} = 0 \qquad \hat{\jmath}\cdot\hat{\jmath} = 1 \qquad \text{etc, etc} \qquad 3.5$$

The dot product of two <u>perpendicular</u> vectors is 0. 3.6

The order of vectors in a dot product doesn't matter. 3.7

The result of a dot-product is a **<u>scalar</u>**. 3.8

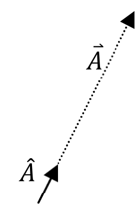
How to Turn Any Vector into a Unit Vector

In physics, a unit vector is a convenient way to express the direction of a force in 3D space. To turn a vector into a unit vector follow these steps:

a) Determine the *magnitude* of the vector using $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$.

b) Divide the original *vector* by its *magnitude*. The unit vector is $\hat{A} = \frac{\vec{A}}{A}$.

The units cancel out & the magnitude of the new *unit* vector $\hat{A}$ is 1.



How to Determine the Angle Between Two Known Vectors

STYLE 1	STYLE 2
$\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{B}$	$\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{B}$
$AB\cos\theta_{AB} = A_xB_x + A_yB_y + A_zB_z$	$AB\cos\theta_{AB} = \vec{A}\cdot\vec{B}$
$\cos\theta_{AB} = \dfrac{A_xB_x + A_yB_y + A_zB_z}{AB}$	$\cos\theta_{AB} = \dfrac{\vec{A}\cdot\vec{B}}{AB}$ **OR** $\hat{A}\cdot\hat{B}$
$\theta_{AB} = \cos^{-1}\left(\dfrac{A_xB_x + A_yB_y + A_zB_z}{AB}\right)$	$\theta_{AB} = \cos^{-1}\left(\dfrac{\vec{A}\cdot\vec{B}}{AB}\right)$ **OR** $\cos^{-1}\left(\hat{A}\cdot\hat{B}\right)$

Angle between a vector and an axis

The above procedure can be used to determine the angle between a vector and any axis.

To determine the angle between $\vec{A}$ and the *positive* x-axis let $\vec{B} = \hat{\imath}$ and do the procedure above.

$$\theta_{pos\ x\ axis} = \cos^{-1}\left(\frac{\vec{A}\cdot\hat{\imath}}{A(1)}\right) = \cos^{-1}\left(\frac{A_x}{A}\right)$$

Another way to derive the above result is to use $\hat{A}\cdot\hat{\imath} = \cos\theta_{pos\ x\ axis}$.

Notice: $\frac{A_x}{A}$ is simply the $\hat{\imath}$ part of $\hat{A}$. We could instead write the above result as

$$\theta_{pos\ x\ axis} = \cos^{-1}\left(\text{the } \hat{\imath} \text{ part of } \hat{A}\right)$$

Tip: if a problem asks for the angle between a vector and the *negative* x-axis we would use

$$\hat{A}\cdot(-\hat{\imath}) = \cos\theta_{neg\ x\ axis}$$

$$\theta_{neg\ x\ axis} = \cos^{-1}\left(-1 \text{ times the } \hat{\imath} \text{ part of } \hat{A}\right)$$

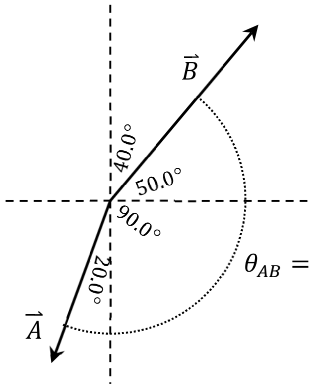
Example of *dot* product for two vectors in *polar* form

Two vectors are $\vec{A} = 3.00 \text{ m @ } 20.0° \text{ west of south}$ and $\vec{B} = 4.00 \text{ m @ } 40.0° \text{ east of north}$.

I want to use the formula $\vec{A} \cdot \vec{B} = AB \cos \theta_{AB}$.

I know $A = \|\vec{A}\| = \text{the magnitude of } \vec{A}$=3.00 m and $B = \|\vec{B}\| = \text{the magnitude of } \vec{B}$=4.00 m.

To get the angle between I draw a quick sketch as shown below.



$$\theta_{AB} = 50.0° + 90.0° + 20.0°$$
$$\theta_{AB} = 160.0°$$

Now I use the formula to get

$$\vec{A} \cdot \vec{B} = AB \cos \theta_{AB}$$
$$\vec{A} \cdot \vec{B} = (3.00 \text{ m})(4.00 \text{ m}) \cos(160.0°)$$
$$\boldsymbol{\vec{A} \cdot \vec{B} = -11.\underline{2}8 \text{ m}^2}$$

Notice: we input two vectors and the output is a scalar.

Here the vectors are pointing in nearly *opposite* directions…the negative result makes sense.

Example of *dot* product for two vectors in *Cartesian* form

Two vectors are $\vec{C} = \left(1.00\hat{\imath} - 2.00\hat{\jmath} + 3.00\hat{k}\right)\text{N}$ and $\vec{D} = \left(0.00\hat{\imath} - 4.00\hat{\jmath} - 2.00\hat{k}\right)\text{m}$.

$$\vec{C} \cdot \vec{D} = C_x D_x + C_y D_y + C_z D_z$$
$$\vec{C} \cdot \vec{D} = \{(1.00)(0.00) + (-2.00)(-4.00) + (3.00)(-2.00)\}\text{N} \cdot \text{m}$$
$$\boldsymbol{\vec{C} \cdot \vec{D} = 2.00 \text{ N} \cdot \text{m}}$$

Notice: with 3D vectors it becomes much more important to trust the math.

3.9 Dot Product Practice

You are given four vectors in the table at right.

a) Determine $\vec{A} \cdot \vec{B}$.

b) Determine $\vec{C} \cdot \vec{D}$.

c) Determine the *unit vector* $\hat{A}$ in Cartesian form.

d) Determine the *unit vector* $\hat{C}$ in Cartesian form.

e) Determine the angle between $\vec{A}$ & $\vec{B}$.

f) Determine the angle between $\vec{C}$ & $\vec{D}$.

g) Determine the angle between $\vec{C}$ and the *positive* z-axis.

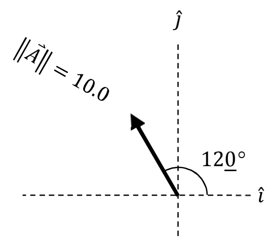
h) Determine $\vec{A} \cdot \vec{C}$.

i) Determine the angle between $\vec{B}$ & $\vec{D}$.

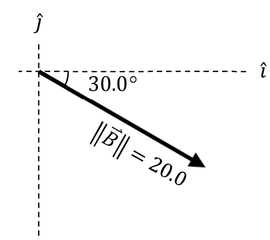
j) Determine the angle between $\vec{D}$ and the *negative* y-axis.

k) Determine the angle between $\vec{B}$ and the *positive* z-axis.

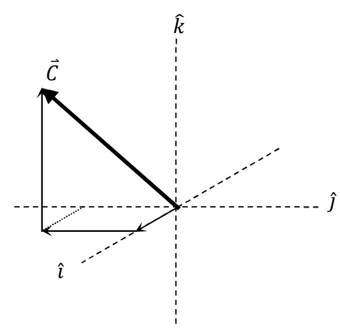
$\vec{A} = 10.0$ directed $12\underline{0}°$ from the positive $x-$axis



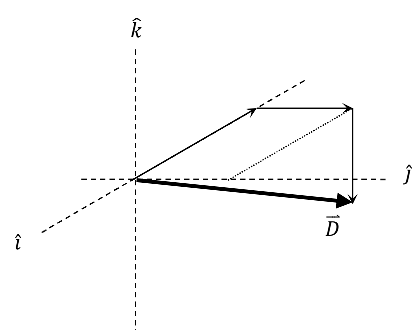
$\vec{B} = 20.0$ directed $-30.0°$ from the positive $x-$axis



$\vec{C} = 1.00\hat{\imath} - 2.00\hat{\jmath} + 3.00\hat{k}$



$\vec{D} = -3.00\hat{\imath} + 1.00\hat{\jmath} - 2.00\hat{k}$



CROSS PRODUCTS

The equations and useful facts for cross-products are given below

$$\vec{A} \times \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) \qquad 3.9$$

The magnitude of the cross-product is $\|\vec{A} \times \vec{B}\| = AB \sin\theta_{AB}$
Note: since θ_{AB} is always between 0° & 180° we know $\sin\theta_{AB} > 0$. 3.10

Use the Right Hand Rule to Determine Direction of Cross-Product

1) Align fingers of right hand with first vector. 3.11
2) Curl fingers of right hand towards second vector.
3) Thumb points in direction of result.

$\hat{\imath} \times \hat{\imath} = 0$ $\quad$ $\hat{\imath} \times \hat{\jmath} = \hat{k}$ $\quad$ $\hat{\jmath} \times \hat{\imath} = -\hat{k}$ $\quad$ $\hat{\jmath} \times \hat{\jmath} = 0$ $\quad$ etc, etc 3.12

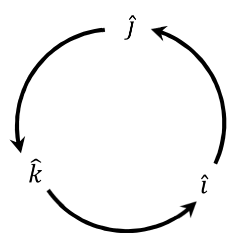
The cross product of two parallel (or anti-parallel) vectors is 0. 3.13

Switching the order of vectors in cross-product flips the sign of the result. 3.14

The magnitude of $\vec{A} \times \vec{B}$ is the area of the parallelogram defined by $\vec{A}$ & $\vec{B}$. 3.15

The result of a cross-product is a **vector**. 3.16

In 3.12, it is difficult to keep track of signs. I remember the signs using a trick I call the wheel of pain (shown at right). You use the wheel of pain to figure out the sign of the cross product between two unit vectors.



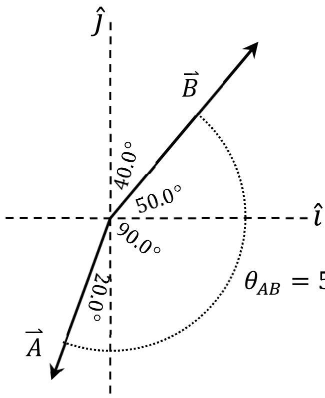
For example, consider $\hat{\imath} \times \hat{k}$. The first term in the cross product is $\hat{\imath}$ so start there in the wheel of pain. The next term is $\hat{k}$ so move around the wheel in that direction. The next term around the wheel is the result (with a ± sign). Here the result of the cross product is $\hat{\imath} \times \hat{k} = -\hat{\jmath}$. The minus sign comes from the fact that we were going opposite the arrows of the wheel of pain.

To practice using the wheel, verify that $\hat{\jmath} \times \hat{k} = \hat{\imath}$ while $\hat{k} \times \hat{\jmath} = -\hat{\imath}$.

Using the wheel of pain to determine cross products is pretty darn fast if one or more of the components in either vector is zero. This is common in physics problems as we are usually free to align our coordinates with one of the vectors and set two of the three vector components to zero.

Example of *cross* product for two vectors in *polar* form

Two vectors are $\vec{A} = 3.00 \text{ m @ } 20.0°$ west of south and $\vec{B} = 4.00 \text{ m @ } 40.0°$ east of north.
I want to use the formula $\|\vec{A} \times \vec{B}\| = AB \sin\theta_{AB}$ to get the magnitude of the cross product.
I must also use the right hand rule to get the direction of the cross product.
Remember: the output of the cross product is a vector so we need both magnitude and direction.
I know $A = 3.00$ m and $B = 4.00$ m.
To get the angle between I draw a quick sketch as shown below.



$$\theta_{AB} = 50.0° + 90.0° + 20.0°$$
$$\theta_{AB} = 160.0°$$

Now I use the formula to get

$$\|\vec{A} \times \vec{B}\| = AB \sin\theta_{AB}$$
$$\|\vec{A} \times \vec{B}\| = (3.00 \text{ m})(4.00 \text{ m}) \sin(160.0°)$$
$$\|\vec{A} \times \vec{B}\| = 0.342 \text{ m}^2$$

Here the vectors are pointing in nearly *opposite* directions…the result close to zero makes sense.
Remember: the *cross* product is zero when the input vectors are parallel (or anti-parallel).

Finally, I use the right hand rule to determine the direction of the cross product.

1) I line up the fingers of my right hand with the first vector in the cross-product (in this case, $\vec{A}$).
2) I curl the fingers of my right hand to the second vector (in this case, curl to $\vec{B}$).
3) I observe the thumb of my right hand points out of the page ($+\hat{k}$ according to this coordinate system).

The final *vector* result of the cross product is thus

$$\boldsymbol{\vec{A} \times \vec{B} = 0.342 \text{ m}^2(+\hat{k})}$$

Example of *cross* product for two vectors in *Cartesian* form using the wheel of pain

Two vectors are $\vec{C} = (1.00\hat{\imath} - 2.00\hat{\jmath} + 3.00\hat{k})\text{N}$ and $\vec{D} = (0.00\hat{\imath} - 4.00\hat{\jmath} - 2.00\hat{k})\text{m}$.

$$\vec{C} \times \vec{D} = (1.00\hat{\imath} - 2.00\hat{\jmath} + 3.00\hat{k})\text{N} \times (0.00\hat{\imath} - 4.00\hat{\jmath} - 2.00\hat{k})\text{m}$$

Note: I know terms with $\hat{\jmath} \times \hat{\jmath} = 0$ and $\hat{k} \times \hat{k} = 0$ will drop out…so I don't even bother to write them.
Also, I take great care to not mess up the order of multiplication…it matters for a *cross* product!

$$\vec{C} \times \vec{D} = \{(1.00\hat{\imath}) \times (-4.00\hat{\jmath}) + (1.00\hat{\imath}) \times (2.00\hat{k}) + (-2.00\hat{\jmath}) \times (-2.00\hat{k}) + (3.00\hat{k}) \times (-4.00\hat{\jmath})\}\text{N·m}$$
$$\vec{C} \times \vec{D} = \{-4.00(\hat{\imath} \times \hat{\jmath}) + 2.00(\hat{\imath} \times \hat{k}) + 4.00(\hat{\jmath} \times \hat{k}) - 12.0(k \times \hat{\jmath})\}\text{N·m}$$

I use the wheel of pain to determine $\hat{\imath} \times \hat{\jmath} = +\hat{k}$, $\hat{\imath} \times \hat{k} = -\hat{\jmath}$, $\hat{\jmath} \times \hat{k} = +\hat{\imath}$, and finally $\hat{k} \times \hat{\jmath} = -\hat{\imath}$.

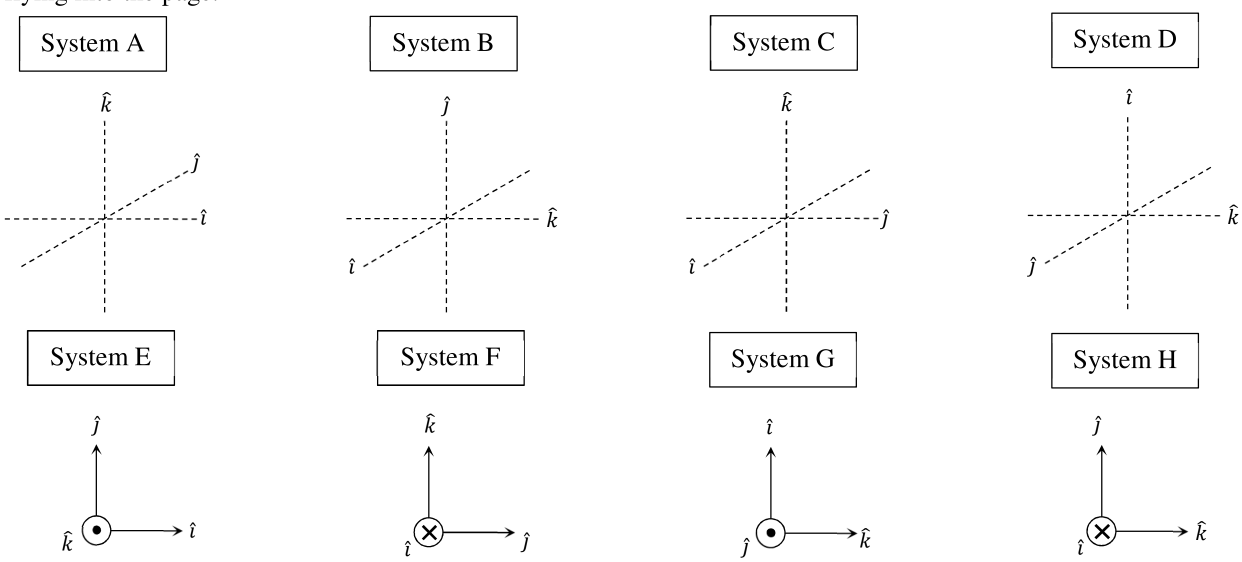
$$\vec{C} \times \vec{D} = \{-4.00(+\hat{k}) + 2.00(-\hat{\jmath}) + 4.00(+\hat{\imath}) - 12.0(-\hat{\imath})\}\text{N·m}$$

Now I group like terms and reorder the sequence in stand format ($\hat{\imath}$ first, then $\hat{\jmath}$, then $\hat{k}$).

$$\boldsymbol{\vec{C} \times \vec{D} = (-8.00\hat{\imath} - 2.00\hat{\jmath} - 4.00\hat{k})\text{N·m}}$$

3.10 Before doing any cross products, you must first learn to recognize if a coordinate system is *right-handed*. A right handed coordinate system is one which conforms to the wheel of pain. To check this, line up the fingers of your right hand with $\hat{\imath}$, curl your fingers to $\hat{\jmath}$, and verify your thumb actually points in the positive $\hat{k}$ direction. If your thumb aligns with the positive $\hat{k}$ direction, the coordinate system is right-handed. Determine which of the following coordinate systems are right handed.

In Systems E through H the $\odot$ symbol implies "out of the page" while the $\otimes$ symbol implies into the page. I remember it this way: the dot is an arrowhead coming out of the page while the × is the feathery tail of the arrow flying into the page.



3.11 Cross Product Practice

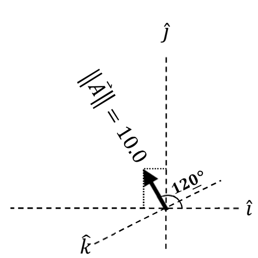
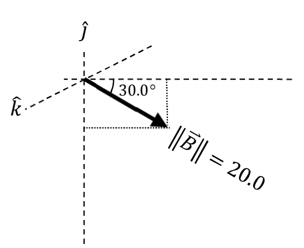
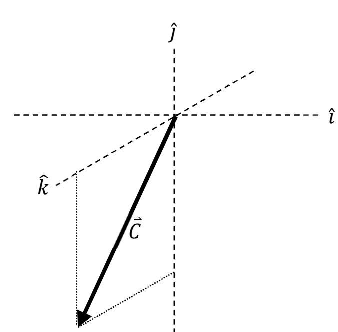
You are given three vectors in the table below. Note: you should immediately verify the coordinate system is right-handed before proceeding. Figures are *approximately* to scale but it isn't perfect.

a) Determine $\vec{A} \times \vec{B}$.

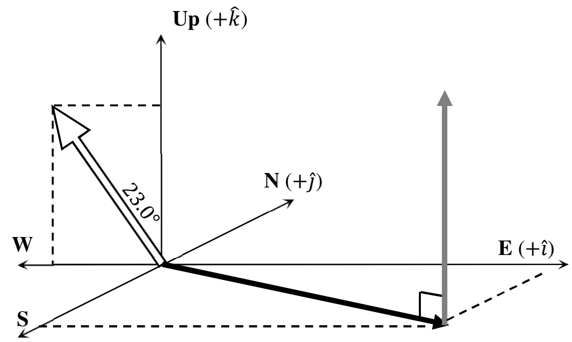
b) Determine $\vec{C} \times \vec{B}$.

c) Why it is possible to compute $\vec{C} \cdot (\vec{A} \times \vec{B})$ but not $\vec{C} \times (\vec{A} \cdot \vec{B})$?

d) Determine $\vec{B} \times \vec{A}$.

$\vec{A} = 10.0$ directed $120°$ from the positive $x-$axis	$\vec{B} = 20.0$ directed $-30.0°$ from the positive $x-$axis	$\vec{C} = 0.00\hat{\imath} - 40.0\hat{\jmath} + 30.0\hat{k}$
		

3.12 A drone, initially at the origin is repositioned using three displacements. The first displacement, shown in black, moves the drone 10.00 m heading 36.87° south of east. The second displacement, shown in grey, moves the drone upwards 7.25 m. The final position of the drone, shown in white, is in the plane defined by the east-west axis and the vertical axis with the angle shown in the figure. The final distance from the origin is 5.00 m.



a) Write down the 1st displacement in Cartesian form with *four* sig figs.
b) Write down the final position vector in Cartesian form with *four* sig figs.
c) What 3rd displacement vector was required to give the drone the stated final position? Answer in Cartesian form with *three* sig figs.

3.13 A position vector is given as $\vec{r} = -3.00\hat{\imath} + 2.00\hat{\jmath}$. You may assume the units on this position vector are meters. A *linear* momentum vector is given by $\vec{p} = 5.00\hat{\imath} - 4.00\hat{k}$. The assumed units on the momentum vector are $\mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}}$. *Angular* momentum ($\vec{L}$) is defined by the equation

$$\vec{L} = \vec{r} \times \vec{p}$$

a) Determine the units appropriate for *angular* momentum.
b) Determine the angle between the position vector and the *linear* momentum vector.
c) Determine the *angular* momentum vector by computing the cross-product. Answer in Cartesian form and keep three sig figs on all components. Don't forget to include the units you determined in part a!!!

3.14 An electric field vector is $\vec{E} = -2.00\hat{\imath} + 3.00\hat{\jmath} - 4.00\hat{k}$. For this problem I will ignore the units. If you care, the units are either $\frac{\mathrm{V}}{\mathrm{m}}$ (volts per meter) or $\frac{\mathrm{N}}{\mathrm{C}}$ (newtons per coulomb).

a) Determine the magnitude of the electric field vector. Tip: when the problem statement is written with decimals, that implies you should answer with as a decimal with the same number of sig figs.
b) Determine a unit vector describing the direction of the electric field. Each term should have a decimal number with three sig figs.
c) Determine the angle between the electric field and the positive z-axis. Again, assume I want three sig figs for final answer.

3.15 The displacement vector between random points **A** and **B** is given by

$$\vec{r} = 2.00\hat{\imath} - 3.00\hat{\jmath} - 4.00\hat{k}$$

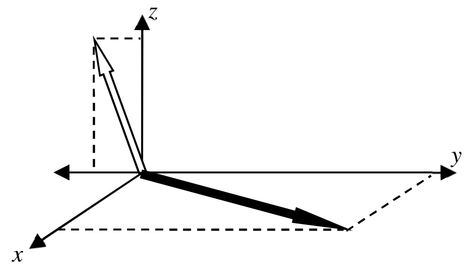
A force vector is

$$\vec{F} = -5.00\hat{\imath} + 6.00\hat{k}$$

Note: the units on $\vec{r}$ are meters (m) and the units on $\vec{F}$ are Newtons (N).

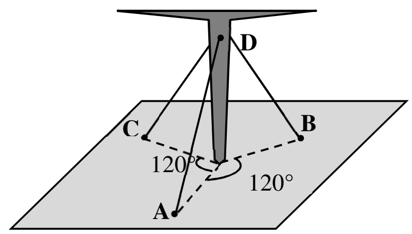
a) Write down a unit vector pointing in the same direction as $\vec{r}$. Answer in Cartesian form with 3 sig figs for each term.
b) Determine the angle between $\vec{r}$ and the positive z-axis. Express your result as a number between 0° and 180°.
c) Torque is defined as $\vec{\tau} = \vec{r} \times \vec{F}$. Determine the torque using the given vectors. Answer in Cartesian form.

3.16 A moth starts from rest and undergoes three displacements. The first displacement is $\vec{A}$ shown in white while the second is $\vec{B}$ shown in black. Vector $\vec{A}$ has magnitude 10 and lies in the yz-plane angled 20° from the z-axis. Vector $\vec{B}$ has magnitude 20 and lies in the xy-plane angled 30° from the y-axis. After a *third* displacement, the moth has returned to the origin.



a) Determine the 3rd displacement in *Cartesian* form.
b) Determine the *magnitude* of the third displacement.
c) Determine the *angle* the third displacement makes with the negative z-axis.

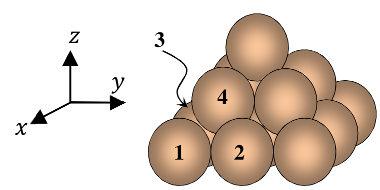
3.17 A power line structure is secured by three guy lines. The three guys are symmetrically located about the base of the structure. Assume the point where the structure touches the ground is the origin. For simplicity, assume all guy lines attach to the structure at the point **D** 20.0 m above the origin. The point **A** lies on the positive x-axis while the axis of the structure lies on the positive z-axis. The positive y-axis is to the right in the figure. The guys are each 35.0 m long.



a) Determine the unit vector $\hat{A}$ pointing from the attachment point to point **A**. Verify $\hat{A}$ is indeed a unit vector by determining its magnitude and showing $\|\hat{A}\| = \sqrt{\hat{A} \cdot \hat{A}} = 1$.
b) Determine the unit vector pointing from the attachment point to point **B**.
c) Determine the unit vector pointing from the attachment point to point **C**.
d) Determine the angle between any two guy lines.
e) **Challenge:** The wind picks up and causes a force of 1000 N that acts at point **D** pointing parallel to the positive y-axis. The tension in guy **C** is 6000 N. The ground pushes straight up (parallel to the z-axis) with unknown magnitude Z. The vector sum of all forces acting on the structure is zero. Determine the tension in guys **A** and **B** and the magnitude of the upwards force from the ground.

This type of computation is could also be used to determine bonding angles in chemistry or in studying crystal lattices in a materials class.

3.18 In a grocery store a bunch of oranges are stacked for display. The bottom plane of oranges in the stack forms a square. Assume each orange has identical radius R. Notice that a coordinate system is shown. In particular note that for *this picture* right and left relate to $\hat{j}$. The center of each of the bottom oranges are in the xy-plane. Some oranges are numbered to ease communication.



a) Determine a vector pointing from of orange 1 to orange 2 (center-to-center).
b) Determine a vector pointing from orange 1 to orange 3 (center-to-center).
c) Determine a vector pointing from orange 1 to orange 4 (center-to-center).

Note: we could change the bottom plane of oranges in the stack to form an equilateral triangle or hexagon. In these orientations we say the oranges are in Face Centered Cubic (FCC) or hexagonal close pack (HCP) respectively. It turns out atoms will pack most efficiently (least amount of empty space) in HCP. Also interesting, FCC and HCP are actually the same lattice structure if one is clever about how you rotate the coordinate system! Lattice structure relates to many properties of materials, for instance electrical conductivity. More on this in Condensed Matter physics courses (previously known as Solid State Physics) or Materials courses.

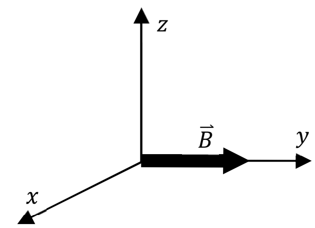
3.19 The quantum mechanical spin of nuclei causes them to act like small magnets. The magnetism is characterized by the magnetic moment $(\vec{\mu})$ which indicates the size and orientation of the magnetism. When exposed to an external magnetic field $(\vec{B})$ the potential energy (U) associated with the molecule in the field is given by

$$U = -\vec{\mu} \cdot \vec{B}$$

The molecule will also experience a torque $(\vec{\tau})$ given by

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

An external magnetic field $\vec{B}$ aligned with the positive *y*-axis has magnitude 1.0 T. A proton at the origin has a magnetic moment with magnitude $\mu = 5.0 \times 10^{-27}\,\frac{\mathrm{J}}{\mathrm{T}}$.



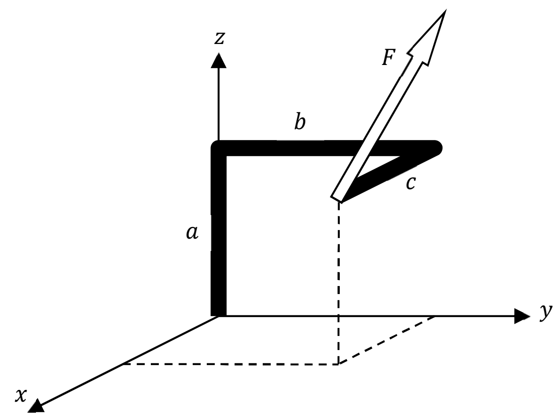
a) For what direction of $\vec{\mu}$ will the proton have <u>max</u> potential energy?
b) For what direction of $\vec{\mu}$ will the proton have <u>min</u> potential energy?
c) Determine the energy difference between the min and the max.
d) For what directions of $\vec{\mu}$ will the proton experience no torque?
e) Determine the potential energy and torque when the magnetic moment is aligned with the positive *z*-axis.
f) Determine all possible orientations of the proton for which it has no potential energy.

Note: In an MRI the magnetic moments align by *precessing* about the external field; they don't point in the same direction but act instead somewhat like a spinning top. While the above example isn't an accurate representation of this, it gets the idea across. Also, expect the numbers in real life to differ from my estimates.

The energy difference between the anti-aligned and aligned states corresponds to radio frequencies. By sending in a pulse of radio waves with a frequency corresponding to the energy from part c) the magnetic moments will flip from aligned to anti-aligned. When the pulse is turned off, the magnetic moments will return to the aligned state. In the process of returning to the lower energy state, the nuclei emit radio waves which can be detected and used to non-destructively determine material composition. There is quite a lot more to this but I thought this would be fun to consider.

Example: Torque is defined as $\vec{\tau} = \vec{r} \times \vec{F}$. The vector $\vec{r}$ is the displacement from the pivot point (axis of rotation) to the point where force is applied.

Consider the figure shown at right. The force of magnitude F lies in a plane parallel to the yz-plane and is applied at the end of the bent, black rod. Assuming the origin is the pivot point, determine the torque exerted by the force on the rod.



$$\vec{r} = c\hat{\imath} + b\hat{\jmath} + a\hat{k}$$
$$\vec{F} = 0\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$
$$\vec{\tau} = \left(c\hat{\imath} + b\hat{\jmath} + a\hat{k}\right) \times \left(F_y\hat{\jmath} + F_z\hat{k}\right)$$

Ignore terms with $\hat{k} \times \hat{k} = 0$ and $\hat{\jmath} \times \hat{\jmath} = 0$.

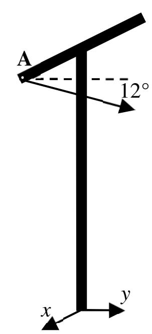
$$\vec{\tau} = cF_y(\hat{\imath} \times \hat{\jmath}) + cF_z\left(\hat{\imath} \times \hat{k}\right) + bF_z\left(\hat{\jmath} \times \hat{k}\right) + aF_y\left(\hat{k} \times \hat{\jmath}\right)$$

Use the wheel of pain to show

$$\vec{\tau} = cF_y\left(\hat{k}\right) + cF_z(-\hat{\jmath}) + bF_z(\hat{\imath}) + aF_y(-\hat{\imath})$$
$$\vec{\tau} = \left(bF_z - aF_y\right)\hat{\imath} - cF_z\hat{\jmath} + cF_y\hat{k}$$

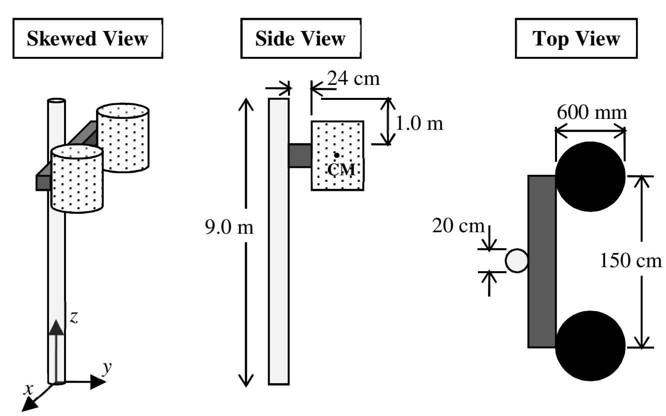
Note: you could figure out F_z and F_y using SOH CAH TOA if an angle is known. I was more interested in showing you how to do the cross-product stuff.

3.20 A simple model of a utility pole is shown at right. The above-ground portion of the pole is approximately 6.00 m tall with a 2.00 m long cross-bar centered on top. The origin is located at the base of the pole which is aligned with the positive z-axis. A wire causes a force of tension at point **A** in the direction indicated with a magnitude of 2750 N. To be clear, the force of tension is in the yz-plane.



a) Determine the position vector of point **A** in Cartesian form.
b) Write the force vector $\vec{F}$ in Cartesian form.
c) The torque caused by the tension is given by $\vec{\tau} = \vec{r} \times \vec{F}$. Determine $\vec{\tau}$.

3.21 Two pole-mount transformers are attached to a utility pole. Doing a quick web search for specifications, I found a transformer with the specs shown in the figure. The cross-rod (dark grey rod) is 24 cm by 24 cm by 1.5 m. The pole is 9.0 m tall with 20 cm diameter. The top of the cross-rod attaches 1.0 m below the top of the pole. Various views shown in an attempt to make the dimensions of the problem easier to see. **Note: figures not to scale.** The center of mass of the transformer is shown in the side view labeled as **CM**.



a) Assume the origin is located at the center of the pole's base. Determine the position vectors ($\vec{r}$) for the center of mass (CM) of each transformer.
b) The transformer's weight is a force with magnitude 3000 N that acts at the center of mass pointing straight down ($-\hat{k}$). The torque exerted by the transformer is given by $\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{F}$is the weight force. Determine $\vec{\tau}$ for each transformer.

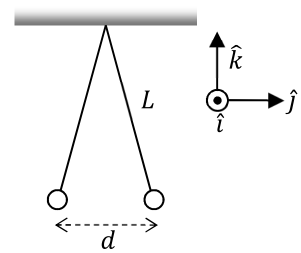
Comments: I tried to use the units I found upon doing a web search. This shows how mixed units might appear in a real world design problem. Notice even a simple problem like this requires attention to detail.

3.22 Two vectors lie in the *xy*-plane. The angle between the two vectors is 153.44° while the dot product of the two vectors is -1000. One of the vectors has a scalar *x*-component of 20 while the other has a scalar *x*-component of -30. Determine the magnitude of each vector.

3.23 A student gives you a puzzle he claims has two solutions. The student identifies two vectors that lie in the *xy*-plane. He says the dot product of the two vectors is -6.000 while the cross product is 8.000. The first vector has magnitude A = 5.000 while the second has a scalar *x*-component of 1.500. He challenges you to determine two possible solutions for $\vec{A}$ and $\vec{B}$ and the angle between them.

3.24 Two balls hang from the ceiling using light strings. Each ball has static charge on it. As a result, the two balls repel from one another and hang at the angle shown in the picture. The distance of separation is d and the length of each string is L. Assume the size of each ball is negligible compared to the length of the string. Assume the origin of the coordinate system is the point where the two strings attach to the ceiling.

a) Determine a vector describing the displacement from the *left* ball to the origin.
b) Determine a vector describing the displacement from the *right* ball to the origin.



3.25 This time *three* balls are hanging from strings. Each ball again has static charge. The balls separate from each other and form an equilateral triangle in the horizontal plane. The side of the equilateral triangle is s and the length of each string is L. Assume the size of each ball is negligible compared to the length of the string. Assume the origin of the coordinate system is the point where the three strings attach to the ceiling.

a) Determine the distance between the center of triangle and any one of the balls.
b) Determine a vector describing the displacement from the *front left* ball to the origin.
c) At what angle, from the vertical axis, does the ball hang?