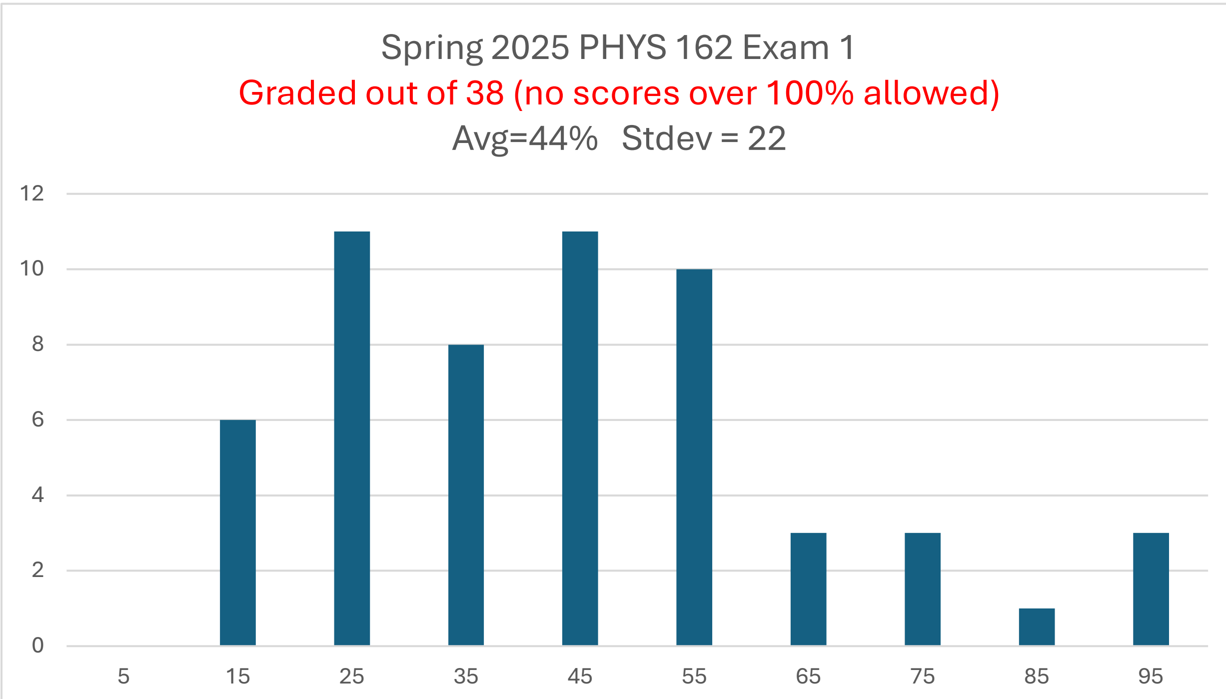
162sp25t1cSoln

Distribution on this page. Solutions begin on the next page.



1a) Note: even though the period has 4 sig figs, it is essentially a 3-sig fig number because the first digit is 1.

$$\omega_0 = \frac{2\pi}{\mathbb{T}} = \frac{2\pi}{1.700\ \text{s}} = \mathbf{3.6\underline{9}6\,\frac{rad}{s}}$$

1b) Use the *unrounded* answer for ω_0 to avoid intermediate rounding error!

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \rightarrow \quad k = m\omega_0^2 = (0.424\ \text{kg})\left(3.6\underline{9}6\,\frac{\text{rad}}{\text{s}}\right)^2 = \mathbf{5.7\underline{9}2\,\frac{N}{m}}$$

1c) While other techniques may be required to determine ϕ, we often set $t = 0$ in $v(t)$ & $x(t)$ and take a ratio.

$$v_i = -\omega_0 A \sin\phi$$

$$x_i = A\cos\phi$$

Taking the ratio gives

$$\frac{v_i}{x_i} = -\omega_0 \tan\phi$$

$$\phi = \tan^{-1}\left(\frac{-v_i}{\omega_0 x_i}\right)$$

$$\phi = \tan^{-1}\left[\frac{-\left(2.00\,\frac{\text{m}}{\text{s}}\right)}{\left(3.6\underline{9}6\,\frac{\text{rad}}{\text{s}}\right)(-0.235\ \text{m})}\right]$$

Notice I converted the units on x_i to meters!

Notice the minus sign on x_i because it was initially to the left of the equilibrium position.

$$\phi = 66.\underline{5}3^\circ \quad \textbf{OR} \quad -113.\underline{4}7^\circ$$

Consider both possible outcomes for the inverse trig function then check which gives the correct signs for x_i & v_i. In this case, $\sin(-113.\underline{4}7^\circ) < 0$ makes $v_i > 0$ as required and $\cos(-113.\underline{4}7^\circ) < 0$ makes $x_i < 0$ as required.

$$\boldsymbol{\phi = -113.\underline{4}7^\circ = -1.98\underline{0}4\ \textbf{rad}}$$

1d) Amplitude of oscillation can be found in several different ways. One could consider an energy equation:

$$K_i + U_{spring_i} = E_{total} = \frac{1}{2}kA^2$$

Alternatively, since we already know the phase angle, it is probably faster to use

$$x_i = A\cos\phi \quad \rightarrow \quad A = \frac{x_i}{\cos\phi} = \frac{-235\ \text{mm}}{\cos(-113.\underline{4}7^\circ)} = \mathbf{59\underline{0}.0\ mm}$$

1e) **WATCH OUT!** Pay close attention to the distinction between degrees and radians when computing! Also, be sure you use unrounded values to avoid intermediate rounding error.

$$x(t) = A\cos(\omega_0 t + \phi)$$

$$x(t) = (59\underline{0}.0\ \text{mm})\cos\left[\left(3.6\underline{9}6\,\frac{\text{rad}}{\text{s}}\right)(1.000\ \text{s}) + (-113.\underline{4}7^\circ)\right]$$

$$x(1.000\ \text{s}) = (59\underline{0}.0\ \text{mm})\cos[3.6\underline{9}6\ \text{rad} - 113.\underline{4}7^\circ]$$

Be sure to convert to equal units and use the correct mode on your calculator!

Since my calculator was already in degrees mode, I found it easiest to convert 3.6$\underline{9}$6 rad to $21\underline{1}.76^\circ$.

$$x(1.000\ \text{s}) = (59\underline{0}.0\ \text{mm})\cos[9\underline{8}.29^\circ]$$

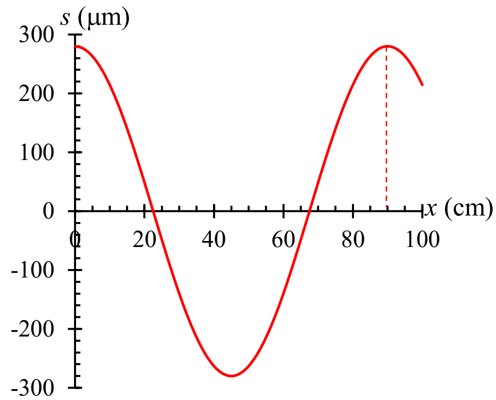
$$x(1.000\ \text{s}) = \mathbf{-85.1\ mm}$$

Since this is my last calculation, no need to keep the unrounded number. Notice the subtraction implies only 2 sig figs should appear on the final result. Our default for exams is to keep three sig figs…

2a) In the graph we see $\lambda = 90.0$ cm.
The standard definition of wavenumber in physics is

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.900\ \mathrm{m}} = \mathbf{6.9\underline{8}1\ m^{-1}}$$

Note: when reading off a graph, it is more realistic to assume we probably get 2 sig figs (not 3). That said, we agreed to default to three sig figs on exam questions.



2b) Period relates to frequency.
Frequency and wavelength relate to wave speed.

$$v_{sound} = f\lambda \quad \rightarrow \quad v_{sound} = \frac{\lambda}{\mathbb{T}} \quad \rightarrow \quad \mathbb{T} = \frac{\lambda}{v_{sound}} = \mathbf{2.5\underline{8}5\ ms}$$

In the real world, we might only assume two sig figs on this result since it came from reading a graph.

2c) In this class, we are assuming the speed of sound relates to air temperature using

$$v_{sound} = 331.4\frac{\mathrm{m}}{\mathrm{s}} + \left(0.61\frac{\mathrm{m}}{\mathrm{s}\cdot{}^\circ\mathrm{C}}\right)T_C$$

$$T_C = \frac{v_{sound} - 331.4\frac{\mathrm{m}}{\mathrm{s}}}{0.61\frac{\mathrm{m}}{\mathrm{s}\cdot{}^\circ\mathrm{C}}}$$

$$\boldsymbol{T_C = 2\underline{7}.4}\ {}^\circ\mathrm{C}$$

Strictly speaking, this approximation is valid for one specific value of air density at standard atmospheric conditions.
That said, the density I chose is *approximately* equal to the density of air those standard conditions.
Our estimate for temperature should be in the right ballpark.

2d) The displacement amplitude in the plot is $s_{max} = 280$ μm.
Maximum overpressure (overpressure amplitude) relates to displacement amplitude using

$$\Delta P_{max} = v_{sound}\rho\omega s_{max}$$

$$\Delta P_{max} = v_{sound}\rho\left(\frac{2\pi}{\mathbb{T}}\right)s_{max}$$

Before plugging in numbers, I thought carefully about the units & prefixes involved.
In particular, watch out for the density conversion!

$$1.215 \times 10^{-3}\frac{\mathrm{g}}{\mathrm{cm}^3} \times \frac{1\ \mathrm{kg}}{1000\ \mathrm{g}} \times \frac{(100\ \mathrm{cm})^3}{(1\ \mathrm{m})^3} = 1.215\frac{\mathrm{kg}}{\mathrm{m}^3}$$

Now plug into value in our formula. Notice I incorporated the prefix from the vertical axis of the plot!

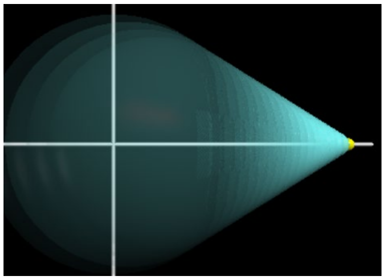
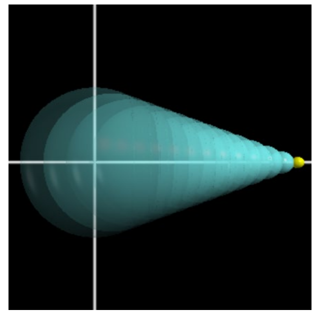
$$\Delta P_{max} = \left(348.1\frac{\mathrm{m}}{\mathrm{s}}\right)\left(1.215\frac{\mathrm{kg}}{\mathrm{m}^3}\right)\left(\frac{2\pi}{0.002585\ \mathrm{s}}\right)(280 \times 10^{-6}\ \mathrm{m})$$

$$\boldsymbol{\Delta P_{max} = 2.878 \times 10^2 = 288\ \mathrm{Pa}}$$

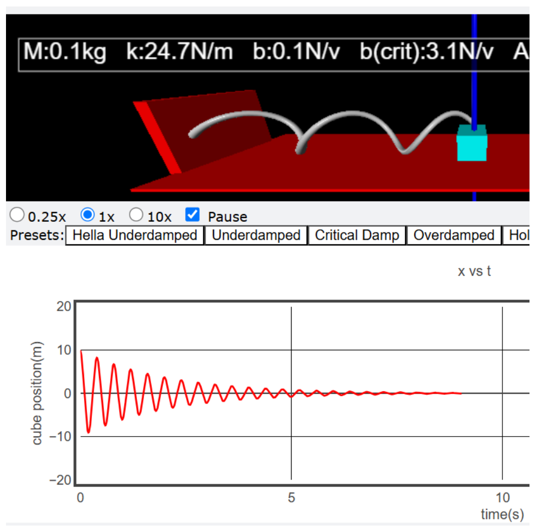
3) **The Mach angle increases.**
Whenever possible, use an equation to guide your reasoning on conceptual questions.

$$\frac{v_{source}}{v_{sound}} = \frac{1}{\sin\theta_{Mach}} \quad \rightarrow \quad \theta_{Mach} = \sin^{-1}\left(\frac{v_{sound}}{v_{source}}\right)$$

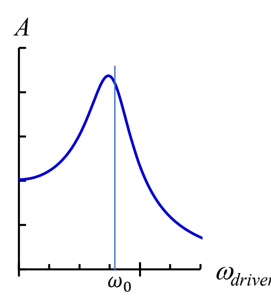
If the jet (the source) slows down, we are taking $\sin^{-1}$ of a larger number which gives a larger angle.

Mach cone at $v_{source} = 2v_{sound}$	**Mach cone at $v_{source} = 3v_{sound}$**
	

4a) As shown in lecture, the "hella underdamped" mass oscillates with gradually decreasing amplitude.



4b) The amplitude of oscillations for a damped-driven system is maximized when $\omega_{driver} \approx \omega_0$ (or $f_{driver} \approx f_0$)
Note: we can convert the angular frequencies to frequencies by dividing both sides by 2π.
It this situation, the resonance frequency is *higher* than the driving frequency.



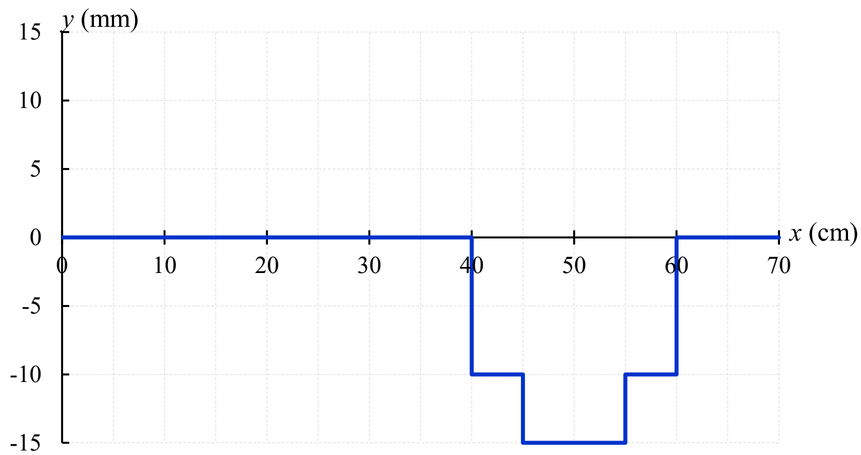
To *decrease* the amplitude of oscillations we must either 1) *decrease the driving frequency* or 2) *increase the resonance frequency* given by

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Decreasing the mass will increase resonance frequency and thus decrease the amplitude of oscillations.
Decreasing the driving frequency was also one of the options provided.

5a) The wave pulse centered at $x = 50$ cm reflects off the right boundary and is re-centered at $x = 50$ cm in 4.00 s.
Note: it inverts as it reflects off the rigid boundary.
Meanwhile, the pulse centered at $x = 10$ cm travels to the right and is centered at $x = 50$ cm in 4.00 s.

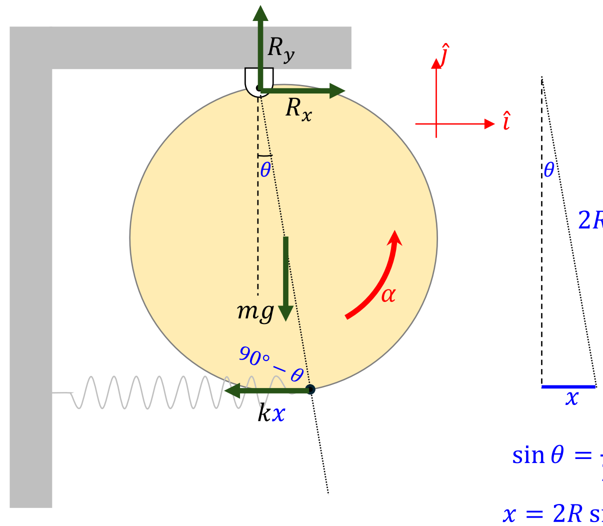
5b) After both pulses are centered at $x = 50$ cm the shape of the string looks like the figure shown below.



6) At right I show forces acting on the *sphere*.
The force components acting at the pivot $(R_x \,\& R_y)$ do not exert torque on the disk (line of action runs through pivot).

I will assume the angular acceleration (α) of the sphere is in the $+\hat{k}$ direction, counter-clockwise (CCW) rotation.

At the instant shown, $\alpha < 0$.
A *negative* α CCW implies a *positive* α clockwise (CW).



$$\sin\theta = \frac{x}{2R}$$

$$x = 2R\sin\theta$$

Using a right hand rule, one sees the torques exerted on the sphere by mg & kx are in the $-\hat{k}$ direction.
Sum of torques gives:

$$\tau_{mg}(-\hat{k}) + \tau_{kx}(-\hat{k}) = I\alpha(+\hat{k})$$

$$-(\tau_{mg} + \tau_{kx}) = I\alpha$$

$$-(Rmg\sin\theta + 2Rkx\sin(90° - \theta)) = I\alpha$$

We can always use $\sin(90° - \theta) = \cos\theta$. Furthermore, in this case the small angle gives $\cos\theta \approx 1 - \frac{\theta^2}{2} \approx 1$.
Using the triangle found on the right side of the above figure we can sub in for x to find

$$-[Rmg\sin\theta + 2Rk(2R\sin\theta)] = I\alpha$$

$$-(Rmg + 4kR^2)\sin\theta = I\alpha$$

For small angle oscillations we may use $\sin\theta \approx \theta$. Furthermore, rewrite using $\alpha = \ddot{\theta}$ and divide both sides by I.

$$-\left(\frac{Rmg + 4kR^2}{I}\right)\theta = \ddot{\theta}$$

When the equation is in this form, we can identify the angular frequency of small oscillations is

$$\omega_0 = \sqrt{\frac{Rmg + 4kR^2}{I}}$$

Finally, notice we must use the parallel axis theorem to determine the moment of inertia of the sphere.

$$I = \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2$$

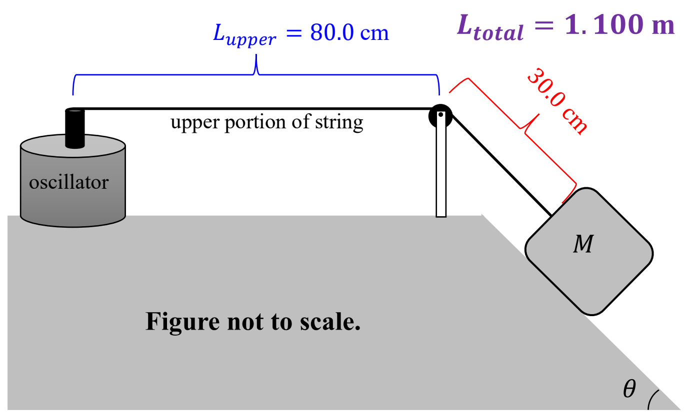
Plugging this into ω_0 gives

$$\omega_0 = \sqrt{\frac{Rmg + 4kR^2}{\frac{7}{5}mR^2}}$$

Multiply the numerator & denominator by 5 to reduce the number of fractions. An R cancels in all terms.

$$\boldsymbol{\omega_0 = \sqrt{\frac{5mg + 20kR}{7mR}}}$$

7) This was my attempt to throw the class a bone for dealing with all the typos. Hopefully you found this problem nearly identical to the lab. In future years, I may give standing wave problems involving ratios of two standing wave states. Be sure you practice problems like **16.18** through **16.22** and **17.12** through **17.15** as well.



For standing wave problems we know

$$v = f_n\lambda_n$$

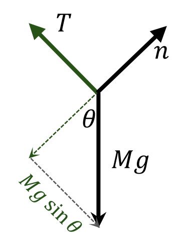
$$\sqrt{\frac{Tension}{\mu}} = f_n\lambda_n$$

The free body diagram (lower figure at right) gives

$$T = Tension = Mg\sin\theta$$

Plugging that in gives

$$\sqrt{\frac{Mg\sin\theta}{\mu}} = f_n\lambda_n$$



Solving for θ algebraically gives

$$\theta = \sin^{-1}\left(\frac{f_n^2\lambda_n^2\mu}{Mg}\right)$$

This standing wave has matched boundary conditions with nodes at each end of the upper portion of the string. In lecture & lab we observed the number of antinodes corresponds to the harmonic number under these conditions. From the problem statement we identify $n = 4$ from knowing there are four antinodes observed.

Because we have a standing wave on the upper portion of the string with matched boundary conditions

$$L_{upper} = n\left(\frac{\lambda_n}{2}\right) \quad \rightarrow \quad \lambda_n = \frac{2L_{upper}}{n} = \frac{2}{4}L_{upper} = 40.0\text{ cm} = 0.400\text{ m}$$

The mass per unit length of the string must incorporate the total length of the string.

$$\mu = \frac{m_{string}}{L_{total}} = \frac{9.25\times10^{-3}\text{ kg}}{1.100\text{ m}} = 8.4\underline{0}9\times10^{-3}\,\frac{\text{kg}}{\text{m}}$$

Plugging everything in gives

$$\theta = \sin^{-1}\left[\frac{(76.4\text{ Hz})^2(0.400\text{ m})^2\left(8.4\underline{0}9\times10^{-3}\,\frac{\text{kg}}{\text{m}}\right)}{(1.000\text{ kg})\left(9.80\,\frac{\text{m}}{\text{s}^2}\right)}\right]$$

$$\boldsymbol{\theta = 53.3°}$$

8a) The distance between the drone and the wall is *decreasing*. The frequency should be shifted *upwards*!

8b) There is a frequency shift of the tone emitted by the drone AND a second shift as the drone measures the echo!

For the shift of the *emitted* wave, the drone acts as the source while the wall (with zero speed) acts as the observer.

$$f' = f_{emitted}\frac{c \pm v_{observer}}{c \mp v_{source}}$$

Use $v_{observer} = 0$ for the wall and use the *negative sign* in the denominator to get an *upwards* frequency shift.

$$f' = f_{emitted}\frac{c}{c - v_{drone}}$$

For the *echo* (reflected wave), the wall acts as the source and the drone acts as the observer.

$$f'' = f'\frac{c \pm v_{observer}}{c \mp v_{source}}$$

We use $v_{source} = 0$ for the wall and must use the *plus sign* in the numerator to get an *upwards* frequency shift.

$$f'' = f'\frac{c + v_{drone}}{c}$$

Now plug in the previously shifted result

$$f'' = \left(f_{emitted}\frac{c}{c - v_{drone}}\right)\frac{c + v_{drone}}{c}$$

$$f'' = f_{emitted}\frac{c + v_{drone}}{c - v_{drone}}$$

Now we can use

$$f'' = f_{emitted} \text{ shifted } upwards \text{ by } 8.50\% = f_{emitted}(1 + 8.50\%) = 1.0850 f_{emitted}$$

Plugging this in gives

$$1.0850 f_{emitted} = f_{emitted}\frac{c + v_{drone}}{c - v_{drone}}$$

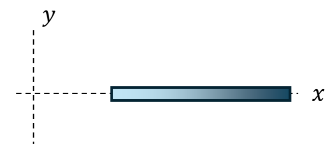
$$1.0850(c - v_{drone}) = c + v_{drone}$$

$$0.0850c = 2.0850 v_{drone}$$

$$v_{drone} = \frac{0.0850}{2.0850}c$$

$$\boldsymbol{v_{drone} = 0.0408c}$$

9a) Linear mass density is given by $\mu(x) = kx^6$.
Positions in the rod with large values of x have large values for μ.
The *right* end of the rod has more mass density.
Wave speed relates to mass density using



$$v = \sqrt{\frac{Tension}{\mu}}$$

The wave speed decreases as it travels from the left end to the right end of the rod.

9b) To get the units I did the following:

$$[\mu] = [k][x^6] \quad \rightarrow \quad [k] = \frac{[\mu]}{[x^6]} = \frac{\frac{\mathrm{kg}}{\mathrm{m}}}{\mathrm{m}^6} = \frac{\mathbf{kg}}{\mathbf{m^7}}$$

Be aware the *units* for the constant k should be represented by $[k]$. If you wrote $k = \frac{\mathrm{kg}}{\mathrm{m}^7}$ I took off half a point. It may not seem like a big deal to you, but to me there is a huge difference between $[k]$, just the units of the parameter, and k, a number with units.

9c) Classic engineering. We know our output for this problem (pulse requires time $\Delta t = 0.500$ s to travel).
We know a procedure to determine Δt for a wave pulse to travel.
Proceed with the standard procedure algebraically, then plug in knowns at the end.

$$v = \frac{dx}{dt}$$

$$dt = \frac{dx}{v}$$

$$\int_{t_i}^{t_f} dt = \int_{x_i}^{x_f} \frac{dx}{v}$$

$$\Delta t = \int_{x_i}^{x_f} \frac{dx}{\sqrt{\frac{Tension}{\mu}}}$$

$$\Delta t = \int_{x_i}^{x_f} \sqrt{\frac{\mu}{Tension}}\, dx$$

$$\Delta t = \int_{x_i}^{x_f} \sqrt{\frac{kx^6}{Tension}}\, dx$$

Recall both tension and the parameter k are constant. Pull those out of the integral & use $\sqrt{x^6} = x^3$.

$$\Delta t = \frac{\sqrt{k}}{\sqrt{Tension}} \int_{x_i}^{x_f} x^3\, dx$$

$$\Delta t = \frac{\sqrt{k}}{\sqrt{Tension}} \left[\frac{x^4}{4}\right]_{x_i}^{x_f}$$

$$\Delta t = \frac{\sqrt{k}}{4\sqrt{Tension}} \left(x_f^4 - x_i^4\right)$$

On the next page I'll solve for k:

From the previous page we know:

$$\Delta t = \frac{\sqrt{k}}{4\sqrt{Tension}}\left(x_f^4 - x_i^4\right)$$

Solve for k:

$$\sqrt{k} = \frac{4\Delta t\sqrt{Tension}}{\left(x_f^4 - x_i^4\right)}$$

$$k = \frac{16(\Delta t)^2 Tension}{\left(x_f^4 - x_i^4\right)^2}$$

$$k = \frac{16(\Delta t)^2 Tension}{\left(x_f^4 - x_i^4\right)^2}$$

WATCH OUT! In this problem $x_i^4 = (1.000\ \mathrm{m})^4 = 1.000\ \mathrm{m}^4$ while $x_f^4 = (3.00\ \mathrm{m})^4 = 81.0\ \mathrm{m}^4$.

$$k = \frac{16(0.500\ \mathrm{s})^2\left(120\underline{0}\ \mathrm{kg}\cdot\frac{\mathrm{m}}{\mathrm{s}^2}\right)}{(81.0\ \mathrm{m}^4 - 1.000\ \mathrm{m}^4)^2}$$

$$\boldsymbol{k = 0.750\,\frac{\mathbf{kg}}{\mathbf{m}^7}}$$

Consider a horizontal mass spring system with negligible friction.
Suppose you know the initial position (x_i), initial velocity (v_i), and period ($\mathbb{T}$).
You do NOT know the mass or the spring constant.
Is it possible to determine the amplitude of oscillations with this information alone?
IF YES, derive an algebraic expression for the amplitude of oscillations in terms of the known parameters.
IF NO, clearly explain why not & state what additional parameter (or parameters) is required to determine amplitude.

It is possible if you relate the parameters using energy methods.

The clever part:
Recall total energy is constant when damping is negligible.
At some instant in time, the object will have $v = 0$.
At this instant, E_{total} equals *maximum possible* spring potential energy $\left(\frac{1}{2}kA^2\right)$.

$$K_i + U_i = E_{total}$$

$$\frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 = \frac{1}{2}kA^2$$

Multiply all terms by $\frac{2}{k}$.

$$\frac{m}{k}v_i^2 + x_i^2 = A^2$$

Recognize $\frac{m}{k}$ relates to the period. Work out that relationship so you can plug it in.

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \rightarrow \quad \mathbb{T} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} \quad \rightarrow \quad \frac{m}{k} = \frac{\mathbb{T}^2}{4\pi^2}$$

Plugging in the result for

$$\frac{\mathbb{T}^2}{4\pi^2}v_i^2 + x_i^2 = A^2$$

Solving for amplitude gives

$$\boldsymbol{A = \sqrt{\left(\frac{v_i\mathbb{T}}{2\pi}\right)^2 + x_i^2}}$$

Several of you approached this by first determining the phase angle.

$$\phi = \tan^{-1}\left(-\frac{v_i}{\omega_0 x_i}\right) = \tan^{-1}\left(-\frac{v_i\mathbb{T}}{2\pi x_i}\right)$$

Then you plugged that into

$$A = \frac{x_i}{\cos\left[\tan^{-1}\left(-\frac{v_i\mathbb{T}}{2\pi x_i}\right)\right]}$$

I was happy to see you came up with a solution and was stoked to give out points!
That said, I hope you agree the energy approach is *significantly* preferable.
Think: the energy approach has a single solution while the phase angle approach requires thinking about both possible solutions caused by the inverse trig function.