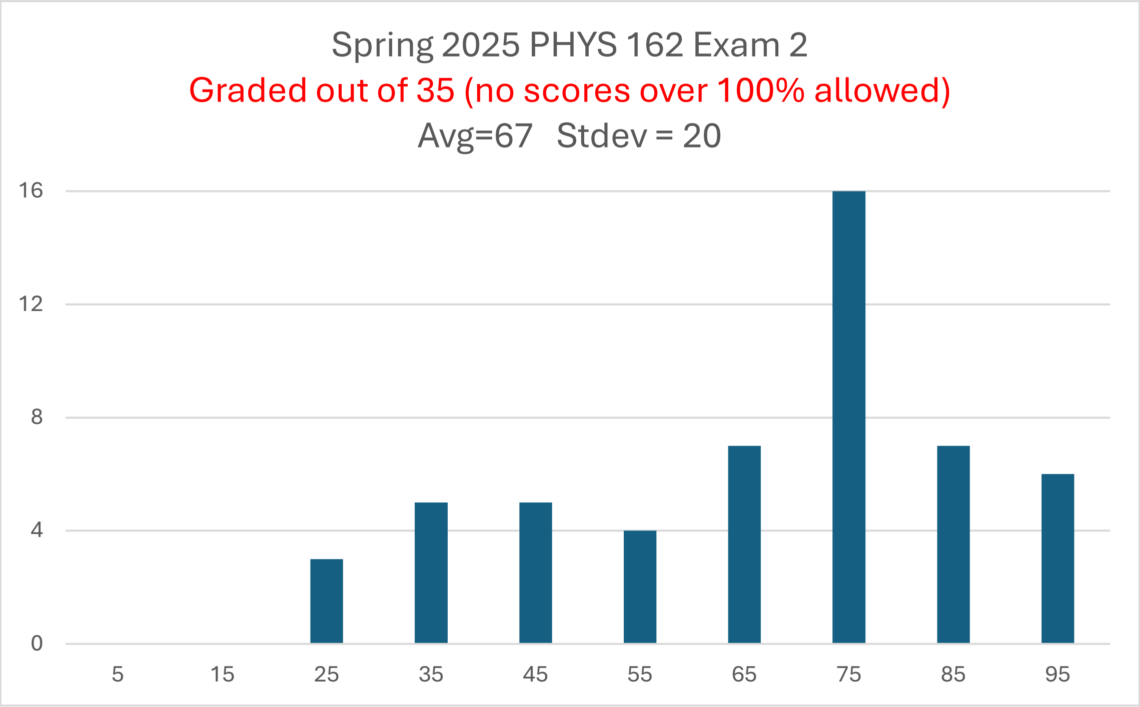
Version A

162sp25t2aSoln

Distribution on this page. Solutions begin on the next page.



Version A

1a) Determine mass using density. **WATCH OUT!** Notice, as often occurs in real life, we have mixed prefixes… For me it seemed easiest to convert the diameter of $4.44\text{ mm} \rightarrow 0.444\text{ cm}$.

$$\rho = \frac{m}{V} \qquad \rightarrow \qquad m = \rho V = \rho\left(\frac{\pi}{4}d^2L\right) = 5.55\frac{\text{g}}{\text{cm}^3}\left[\frac{\pi}{4}(0.444\text{ cm})^2(77.7\text{ cm})\right] = \mathbf{66.\underline{7}7\ g}$$

1b) Heat relates to specific heat using

$$Q = mc\Delta T$$

$$c = \frac{Q}{m\Delta T}$$

WATCH OUT! It is reasonable to write the answer in the units provided in the equation sheet $\left(\frac{\text{J}}{\text{kg}\cdot\text{K}}\right)$. This implies we should use mass in kg and heat in J. Fortunately, $\Delta T = 79.9\ ^\circ\text{C} = 79.9\text{ K}$.

$$c = \frac{6.66 \times 10^3\text{ J}}{(0.066\underline{7}7\text{ kg})(77.9\text{ K})}$$

$$\boldsymbol{c = 1248\frac{\mathbf{J}}{\mathbf{kg\cdot K}}}$$

1c) **WATCH OUT!** Notice, as often occurs in real life, we have mixed prefixes…

$$\Delta L = L_0\alpha\Delta T$$

$$\alpha = \frac{\Delta L}{L_0\Delta T}$$

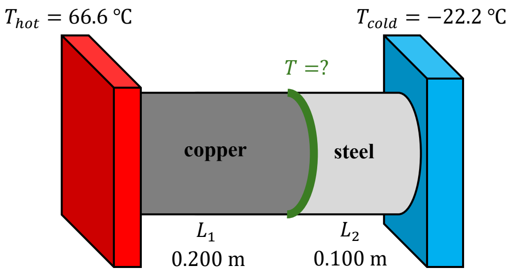
$$\alpha = \frac{0.888 \times 10^{-3}\text{ m}}{(0.777\text{ m})(79.9\ ^\circ\text{C})}$$

$$\alpha = 1.430 \times 10^{-5}\frac{1}{^\circ\text{C}}$$

In more standard units we write

$$\boldsymbol{\alpha = 14.30 \times 10^{-6}\frac{1}{^\circ\mathbf{C}}}$$

Figure not to scale.



2a) Heat generally flows from hot to cold.

There can be exceptions for systems with very small numbers of molecules.

In this problem, heat flows *to the right*.

2b) In steady state, the heat travelling trough the rod 1 (per unit time) must equal heat flowing through rod 2 (per unit time).

If this were not the case, energy would build up at the junction (eventually melting the rods at the junction). This does not happen in the real world.

Therefore, in steady state:

$$\mathcal{P}_1 = \mathcal{P}_2$$

$$\frac{k_1 A_1 \Delta T_1}{L_1} = \frac{k_2 A_2 \Delta T_2}{L_2}$$

The rods have equal areas…cancel area from each side of the equation.

$$\frac{k_1 \Delta T_1}{L_1} = \frac{k_2 \Delta T_2}{L_2}$$

Notice this implies

$$\Delta T_1 = \frac{k_2 L_1}{k_1 L_2} \Delta T_2$$

At this stage many of you tried to use something like the following

$$T_{hot} - T_J = \frac{k_2 L_1}{k_1 L_2} (T_J - T_{cold})$$

Think: to have equal power, the power equations must have the same sign. Think carefully about how you compute ΔT's! Many of you lost points because you computed ΔT with an incorrect sign.

Notice I am using subscript 1 for copper and subscript 2 for steel. To save time I will plug in numbers now…

$$\Delta T_1 = \frac{\left(50 \frac{\mathrm{W}}{\mathrm{m \cdot K}}\right)(0.200\ \mathrm{m})}{\left(400 \frac{\mathrm{W}}{\mathrm{m \cdot K}}\right)(0.100\ \mathrm{m})} \Delta T_2 = 0.25 \Delta T_2$$

Now relate the temperature difference across each rod to total temperature difference between the reservoirs.

$$\Delta T_{total} = \Delta T_1 + \Delta T_2$$

$$\Delta T_{total} = 0.25 \Delta T_2 + \Delta T_2$$

$$\Delta T_{total} = 1.25 \Delta T_2$$

$$\Delta T_2 = \frac{1}{1.25} \Delta T_{total}$$

$$\Delta T_2 = \frac{1}{1.25} [66.6\ ℃ - (-22.2\ ℃)]$$

$$\Delta T_2 = 71.\underline{0}4\ ℃$$

This tells us the temperature of the junction is $71.\underline{0}4$ ℃ larger than the cold reservoir.

$$T_{junction} = 48.\underline{8}4\ ℃$$

2b) To get diameter, know cross-sectional area of a cylinder is $\pi r^2 = \frac{\pi}{4} d^2$ and $\mathcal{P} = \frac{Q}{\Delta t}$. Convert time to seconds.

$$\mathcal{P}_2 = \frac{k_2 A_2 \Delta T_2}{L_2}$$

$$\frac{\pi}{4} d^2 = \frac{L_2 Q}{k_2 \Delta T_2 \Delta t}$$

$$d = \sqrt{\frac{4 L_2 Q}{\pi k_2 \Delta T_2 \Delta t}}$$

$$d = \sqrt{\frac{4(0.100\text{ m})(555\text{ J})}{\pi\left(50\frac{\text{W}}{\text{m}\cdot\text{K}}\right)(71.\underline{0}4\text{ °C})(900\text{ s})}}$$

$$\boldsymbol{d = 4.70\text{ mm}}$$

3a) In this question there is no real need to specify if I mean power emitted or NET power emitted (see extra credit question 1). That said, probably wise to specify if you mean power emitted or NET power emitted when you do these kinds of calculations. I will compute NET power radiated.

$$\mathcal{P}_{NET} = \sigma A_{surface} e\left(T^4 - T_{surroundings}^4\right)$$

$$\mathcal{P}_{NET} = \sigma 4\pi r^2 e\left(T^4 - T_{surroundings}^4\right)$$

$$\mathcal{P}_{NET} = \sigma \pi d^2 e\left(T^4 - T_{surroundings}^4\right)$$

WATCH OUT! This equation does *not* use ΔT! In particular $\left(T^4 - T_{surroundings}^4\right) \neq \Delta T^4$.

This implies we must use the Kelvin scale for temperature!

Notice I also converted the diameter to meters and looked up the emissivity on the exam equation sheet.

$$\mathcal{P}_{NET} = \left(5.67 \times 10^{-8} \frac{\mathrm{W}}{\mathrm{m^2 \cdot K^4}}\right) \pi (0.0750\ \mathrm{m})^2 (0.61) \left[\left(119\underline{3}.15\ \mathrm{K}\right)^4 - \left(293.\underline{1}5\ \mathrm{K}\right)^4\right]$$

$$\mathcal{P}_{NET} = 1\underline{2}34\ \mathrm{W}$$

3b) We know power and time. This will give us a decent estimate for heat. Notice: Q is *negative* when cooling…

$$\mathcal{P} = \frac{Q}{\Delta t} \quad \rightarrow \quad Q = -\mathcal{P}\Delta t = -(1\underline{2}34.2\ \mathrm{W})(0.333\ \mathrm{s}) = -4\underline{1}1.0\ \mathrm{J}$$

Now use this equation in

$$Q = mc\Delta T$$

$$\Delta T = \frac{Q}{mc}$$

$$\Delta T = \frac{-4\underline{1}1.0\ \mathrm{J}}{(1.875\ \mathrm{kg})\left(380 \frac{\mathrm{J}}{\mathrm{kg \cdot K}}\right)}$$

$$\boldsymbol{\Delta T = -0.5\underline{7}7}\ ^{\circ}\mathrm{C}$$

Think: this estimate assumed power was radiated at constant rate.
Strictly speaking, this assumption is invalid when temperature is changing.
HOWEVER, because our time interval is so short, the change in temperature is quite small.
Our assumption of constant rate of power radiated is really quite good.

To get a feeling for the quality of this approximation
I recomputed power radiated using ΔT to estimate the *average* temp $T_{avg} = \left(119\underline{3}.15\ \mathrm{K} - \frac{1}{2} 0.577\ \mathrm{K}\right) = 119\underline{2}.9\ \mathrm{K}$.
I found $\Delta T = -0.5\underline{7}6$ °C.
Because our emissivity has only 2 sig figs, this result has not even changed in the rounding digit!
Furthermore, corrections due to convection and conduction are probably significantly larger than errors associated with our constant temperature assumption.

4) We are asked to the average value of v^3 for the Maxwell-Boltzmann speed distribution.
Because no dimensionality was specifically mentioned, use the default (3D) Maxwell-Boltzmann distribution:

$$f(v) = \left(\frac{m}{2\boldsymbol{\pi} k_B T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_BT}\right) 4\pi v^2$$

Use our standard procedure we have seen several times now:

$$\overline{v^3} = \int_0^\infty v^3 \left(\frac{m}{2\boldsymbol{\pi} k_B T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_BT}\right) 4\pi v^2 \, dv$$

$$\overline{v^3} = 4\pi \left(\frac{a}{\boldsymbol{\pi}}\right)^{3/2} \int_0^\infty v^5 \exp(-av^2) \, dv$$

Identify we should use integral

$$I_5 = \int_0^\infty x^5 e^{-ax^2} dx = \frac{1}{a^3} \quad \text{where} \quad a = \frac{m}{2k_BT}$$

$$\overline{v^3} = 4\pi \left(\frac{a}{\boldsymbol{\pi}}\right)^{3/2} \frac{1}{a^3}$$

If you stopped here, you lost points.
It is unreasonable for students to leave a result in un-simplified form!
You were also told on the exam page to simplify your result.

$$\overline{v^3} = \frac{4}{\pi^{1/2} a^{3/2}}$$

$$\overline{v^3} = \frac{4}{\sqrt{\pi a^3}}$$

If you stopped here, you lost points.
It is standard form to write final answers in terms $m, k_B, \& \ T$.

$$\overline{v^3} = \frac{4}{\sqrt{\pi \left(\frac{m}{2k_BT}\right)^3}} = 4\sqrt{\frac{2^3 k_B^3 T^3}{\pi m^3}}$$

If you stopped here, you lost points.
It is standard form to write this final answer as a single fraction under the radical.

$$\boldsymbol{\overline{v^3}} = \sqrt{\frac{\mathbf{128} \boldsymbol{k_B^3 T^3}}{\boldsymbol{\pi m^3}}}$$

In most engineering applications the 128 & π would be computed as a decimal and brought outside the root.

$$\boldsymbol{\overline{v^3}} \approx \mathbf{6.38} \sqrt{\frac{\boldsymbol{k_B^3 T^3}}{\boldsymbol{m^3}}} = \mathbf{6.38} \left(\frac{\boldsymbol{k_B T}}{\boldsymbol{m}}\right)^{\frac{3}{2}}$$

I'll accept any of these last forms as they all seem pretty clean and easy for a reader to use.

5a) Answer: **REMAIN CONSTANT**
During any isothermal process, temperature remains constant.

$$\Delta E_{int} = nC_V\Delta T = 0$$

5b) Answer: **POSITIVE**
We are told it is an expansion process.
For an expansion process $(V_f > V_i)$, we know W_{by} should be *positive*.
Alternatively, consider

$$\underset{isothermal}{W_{by}} = nRT\ln\frac{V_f}{V_i}$$

Notice the ln term is positive whenever $V_f > V_i$ and negative whenever $V_i > V_f$.

5c) Answer: **POSITIVE**
To understand heat, I will write down the first law then think about W_{by}.

$$\Delta E_{int} = Q_{in} - W_{by}$$

For an isothermal process, we already know $\Delta E_{int} = 0$. This implies:

$$Q_{in} = W_{by}$$

Since we already determined W_{by} is positive, we now know Q_{in} must also be positive.

5d) Answer: **POSITIVE**
Entropy change is given by

$$\Delta S = \int \frac{dQ}{T}$$

For an isothermal process, temperature is constant and we can factor that out of the integral.

$$\Delta S = \frac{1}{T}\int dQ = \frac{Q}{T}$$

Since Q & T are both positive, entropy change is also positive.
Probably didn’t need to do the entire integral to see that, but it is good practice.

6) To derive work for a process, get an expression for P in terms of V.
In this case, the problem statement gave it to us as $P = \alpha V^2$.

$$W_{by} = \int_i^f P\, dV$$

$$W_{by} = \int_i^f \alpha V^2\, dV$$

$$W_{by} = \frac{\alpha}{3} V^3$$

$$W_{by} = \frac{\alpha}{3}\left(V_f^3 - V_i^3\right)$$

If you stopped here, I gave you 2 points out of 3.
This is pretty good, but we should go one step further. Once we are finished you will see why.
Furthermore, the problem statement asked to write this in terms of $P_i, V_i, P_f, \& \ V_f$.
The trick is to rewrite the constant in two ways.
This is exactly the same trick used in the derivation of work done by the gas for an adiabatic process.

$$\alpha = \frac{P}{V^2} = \frac{P_i}{V_i^2} = \frac{P_f}{V_f^2}$$

$$W_{by} = \frac{\alpha}{3} V_f^3 - \frac{\alpha}{3} V_i^3$$

$$W_{by} = \frac{1}{3}\left(\frac{P_f}{V_f^2}\right) V_f^3 - \frac{1}{3}\left(\frac{P_i}{V_i^2}\right) V_i^3$$

$$W_{by} = \frac{1}{3} P_f V_f - \frac{1}{3} P_i V_i$$

$$\boldsymbol{W_{by} = \frac{1}{3}\left(P_f V_f - P_i V_i\right)}$$

Think: if you had to use this equation after getting numbers form a PV diagram, isn't this last form MUCH nicer?

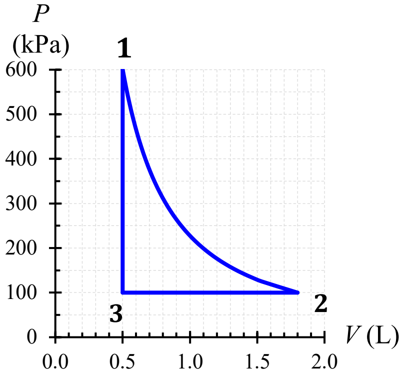
Side note: many of you tried to use $PV = nRT$. While this is true for all gases in our classes, it isn't useful in this unusual process. We were told $P = \alpha V^2$ was constant. Notice this necessarily implies

$$\alpha V^3 = nRT$$

Clearly temperature is NOT constant as the volume changes for this process.
Converting P using the ideal gas law does not make positive progress in terms of solving the integral.

Version A

The cycle starts at point **1** and runs clockwise. **Note: $P_1 = 601$ kPa…*not* 600.** Temperature at point **2** is 650.2 K. Work is done by the gas during every cycle.



7a) Use $PV = nRT$ on point 2. Include units!

7b) Notice process 1 → 2 is *not* isothermal!

$$\frac{P_1V_1}{P_2V_2} = \frac{nRT_1}{nRT_2} \quad \rightarrow \quad T_1 = \frac{P_1V_1}{P_2V_2}T_2$$

7c) Determine the best value to use for degrees of freedom based off discussions in class & homework.

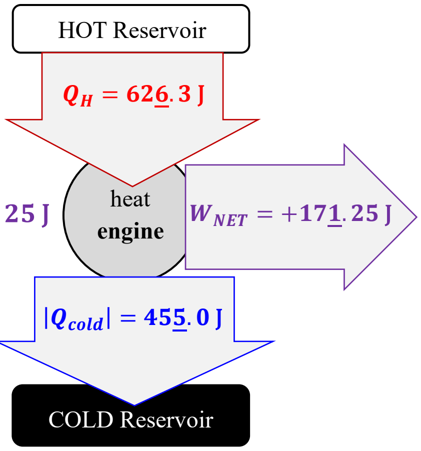
7a	$n = 0.033\underline{3}0$ mol
7b	$T_1 = 10\underline{8}5.5$ K $T_3 = 18\underline{0}.61$ K
7c	$\boldsymbol{f = 5} \quad \rightarrow \quad \gamma = 1.4$ $C_V = \frac{5}{2}R \approx 20.785 \frac{\text{J}}{\text{mol}\cdot\text{K}}$ $C_P = \frac{7}{2}R \approx 29.099 \frac{\text{J}}{\text{mol}\cdot\text{K}}$

In big chart, first fill in any known zeros.

Then fill in using the equation sheet for W_{by} or Q_{in} as needed. Recall one can always use $\Delta E_{int} = nC_V\Delta T$

One can always use the first law for any row: $\Delta E_{int} = Q_{in} - W_{by}$.

	Process Name	Q_{in} (J)	$W_{by\ gas}$ (J)	ΔE_{int} (J)
1 → 2	**Adiabatic** Verify $P_1V_1^\gamma = 22\underline{7}.74$ $P_2V_2^\gamma = 22\underline{7}.71$	0	$W_{by} = \frac{1}{1-\gamma}(P_fV_f - P_iV_i)$ $\mathbf{30\underline{1}.25}$	$\Delta E_{int} = nC_V\Delta T$ $\mathbf{-30\underline{1}.25}$ Verified using 1st Law
2 → 3	**Isobaric**	$Q = nC_P\Delta T = \frac{f+2}{2}nR\Delta T$ $\mathbf{-45\underline{5}.0}$	$W_{by} = P\Delta V$ $\mathbf{-13\underline{0}.0}$	$\Delta E_{int} = nC_V\Delta T$ $\mathbf{-32\underline{5}.0}$ Verified using 1st Law
3 → 1	**Isochoric**	$Q = nC_V\Delta T = \frac{f}{2}nR\Delta T$ $\mathbf{+62\underline{6}.3}$	0	$\Delta E_{int} = nC_V\Delta T$ $\mathbf{+62\underline{6}.3}$ Verified using 1st Law
For the Entire Cycle		Sum the entries in this column $\mathbf{+17\underline{1}.3}$	Sum the entries in this column $\mathbf{+17\underline{1}.25}$	0 Verified using 1st Law



The *upper* arrow uses Q_{hot} = sum all *positive* $Q_{in}'s$ = $\mathbf{62\underline{6}.3\ J}$

The *middle* arrow uses $W_{NET} = \mathbf{+17\underline{1}.25\ J}$

The *lower* arrow uses $|Q_{cold}|$ = |sum all *negative* $Q_{in}'s$| = $\mathbf{45\underline{5}.0\ J}$

$$\textbf{Efficiency} = 1 - \frac{|Q_{cold}|}{Q_{Hot}} = 27.\underline{3}5\%$$

EC1: Consider the equations

Radiative Emission	Radiative Absorption	NET Radiation = Emission-Absorption
$\mathcal{P}_{emit} = \sigma A e T^4$	$\mathcal{P}_{abs} = \sigma A e T_{env}^4$	$\mathcal{P}_{NET} = \sigma A e (T^4 - T_{env}^4)$

If we assume absorption is a 1% correction:

$$0.01\mathcal{P}_{emit} > \mathcal{P}_{abs}$$

$$0.01\sigma A e T^4 > \sigma A e T_{env}^4$$

$$0.01T^4 > T_{env}^4$$

$$T > 3.16T_{env}$$

If we set $T_{env} \approx T_{room} = 293$ K, the absorption term is negligible whenever temperatures exceed

$$T \approx 926 \text{ K} = 633\ ℃$$

The situation in problem 3 far exceeds this minimum temperature.

Furthermore, the sphere's weight is necessarily supported by something (e.g. a wire mesh cooling rack). Conduction of heat directly into this cooling rack is a larger correction than ignoring absorption from the room at these high temperatures. Similarly, convection currents generated in the air immediately surrounding the sphere likely play a larger role than ignoring absorption from the room.

EC2: The *approximate* result is found using the *average* temperature of the sphere as it cools to compute power.

$$T_{avg} = \frac{119\underline{3}.15\ \mathrm{K} + (273.15 + 50.0)\ \mathrm{K}}{2} = 75\underline{8}.15\ \mathrm{K}$$

$$\mathcal{P}_{avg\ NET} = \left(5.67 \times 10^{-8} \frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{K}^4}\right) \pi(0.0750\ \mathrm{m})^2 (0.61) \left[\left(75\underline{8}.15\ \mathrm{K}\right)^4 - \left(293.\underline{1}5\ \mathrm{K}\right)^4\right]$$

$$\mathcal{P}_{avg\ NET} = 197.4\ \mathrm{W}$$

The heat required to cool all the way down to 50.0 °C is

$$Q = mc\Delta T = (1.875\ \mathrm{kg})\left(380 \frac{\mathrm{J}}{\mathrm{kg} \cdot \mathrm{K}}\right)\left(119\underline{3}.15\ \mathrm{K} - 323.15\ \mathrm{K}\right) = 6.199 \times 10^5\ \mathrm{J}$$

This gives

$$\Delta t \approx \frac{Q}{\mathcal{P}_{avg\ NET}} = 3.\underline{1}4 \times 10^3\ \mathrm{s} \approx 0.87\ \mathrm{hr}$$

I will ignore units while deriving the *exact* result found using

$$\mathcal{P} = \frac{dQ}{dt}$$

$$\mathcal{P}\ dt = mc\ dT$$

$$(5.67 \times 10^{-8})\pi(0.0750\ )^2(0.61)[T^4 - T_{env}^4]dt = (1.875)(380)\ dT$$

$$6.112 \times 10^{-10}(T^4 - 7.385 \times 10^9)\ dt = 712.5\ dT$$

$$dt = \frac{712.5}{6.112 \times 10^{-10}(T^4 - T_{env}^4)}\ dT$$

$$dt = \frac{1.1657 \times 10^{12}\ dT}{T^4 - T_{env}^4}$$

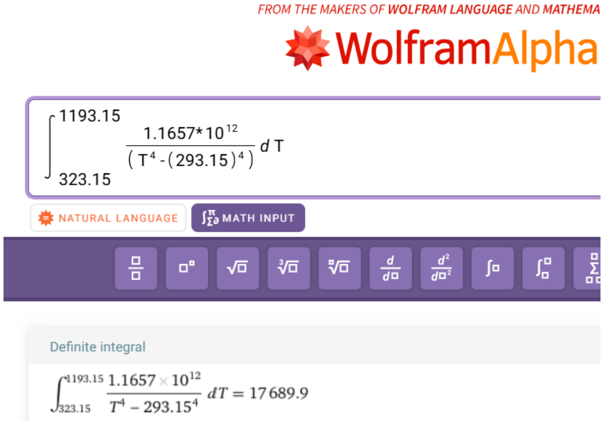
You could approach this integral using partial fractions

$$dt = \frac{1.1657\ dT}{T^4 - T_{env}^4} = 1.1657 \frac{1}{(T^2 - T_{env}^2)(T^2 + T_{env}^2)}\ dT = 1.1657 \frac{1}{(T - T_{env})(T + T_{env})(T^2 + T_{env}^2)}\ dT$$

I'm going to use Wolfram Alpha since I doubt anyone actually tries partial fractions and succeeds.

$$\Delta t \approx \frac{Q}{\mathcal{P}_{avg\ NET}} = 17690\ \mathrm{s} \approx 4.91\ \mathrm{hr}$$

This result seems *a lot* more plausible.
The initial cooling happens quickly ($\mathcal{P}_i \approx 1234$ W).
However, as the sphere asymptotically cools to the final temperature, the rate of cooling drops dramatically.
For example, once the sphere reaches 100 °C = 373 K the cooling rate drops to $\mathcal{P} \approx 7.37$ W.



$$\int_{323.15}^{1193.15} \frac{1.1657*10^{12}}{(T^4-(293.15)^4)}\,d\,T$$

Definite integral

$$\int_{323.15}^{1193.15} \frac{1.1657 \times 10^{12}}{T^4 - 293.15^4}\,dT = 17689.9$$

When one considers conduction and convection, the actual result probably lies somewhere between these two results. I would wait *several* hours before trying to pick up that sphere.

EC3: Consider the equations in the chart. Note: $T_{hot} = T_1$ and $T_{cold} = T_3$.

	Process Name	Q_{in} (J)	$W_{\boldsymbol{by\ gas}}$ (J)	ΔE_{int} (J)
1 → 2	Adiabatic	0	$W_{by} = \frac{1}{1-\gamma}\left(P_f V_f - P_i V_i\right)$	$\Delta E_{int} = nC_V \Delta T$
2 → 3	Isobaric	$Q_{cold} = nC_P \Delta T < 0$	$W_{by} = P\Delta V$	$\Delta E_{int} = nC_V \Delta T$
3 → 1	Isochoric	$Q_{hot} = nC_V \Delta T > 0$	0	$\Delta E_{int} = nC_V \Delta T$

$$Q_{hot} = nC_V\Delta T = \frac{f}{2}nR\Delta T = \frac{f}{2}(P_1V_1 - P_3V_3) = \frac{f}{2}(P_1 - P_3)V_3$$

$$|Q_{cold}| = |nC_P\Delta T| = \left|\frac{f+2}{2}nR\Delta T\right| = \left|\frac{f+2}{2}(P_3V_3 - P_2V_2)\right| = \left|\frac{f+2}{2}P_3(V_3 - V_2)\right| = \frac{f+2}{2}P_3(V_2 - V_3)$$

$$\eta = 1 - \frac{|Q_{cold}|}{Q_{Hot}}$$

$$\eta = 1 - \frac{\frac{f+2}{2}P_3(V_2 - V_3)}{\frac{f}{2}(P_1 - P_3)V_3}$$

$$\eta = 1 - \frac{f+2}{f}\cdot\frac{P_3(V_2 - V_3)}{(P_1 - P_3)V_3}$$

$$\eta = 1 - \gamma\cdot\frac{\frac{V_2}{V_3} - \frac{V_3}{V_3}}{\frac{P_1}{P_3} - \frac{P_3}{P_3}}$$

$$\eta = 1 - \gamma\cdot\frac{\frac{V_2}{V_3} - 1}{\frac{P_1}{P_3} - 1}$$

$$\frac{V_2}{V_3} = compression\ ratio = r_c$$

Using the ideal gas law for the isochoric process gives:

$$\frac{P_1V_1}{P_3V_3} = \frac{nRT_1}{nRT_3} \quad \rightarrow \quad \frac{P_1}{P_3} = \frac{T_1}{T_3} = \frac{T_{hot}}{T_{cold}}$$

$$\boldsymbol{\eta = 1 - \gamma\cdot\frac{r_c - 1}{\frac{T_{hot}}{T_{cold}} - 1}}$$

We could probably write this in other forms, but this seems like a useful set of parameters to use.