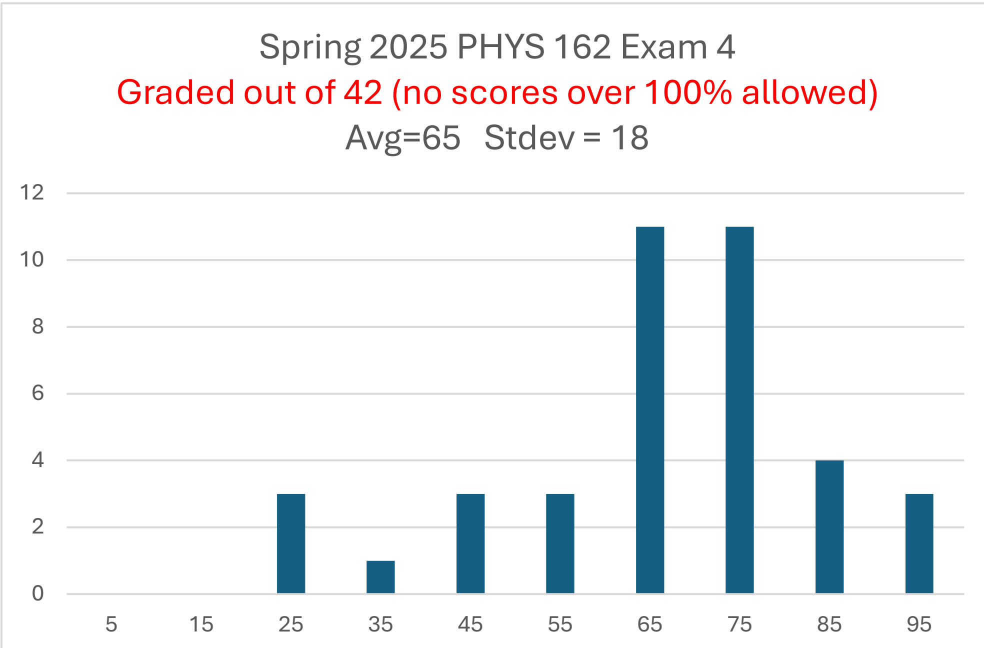
162sp25t4aSoln

Distribution on this page.
Solutions begin on the next page.



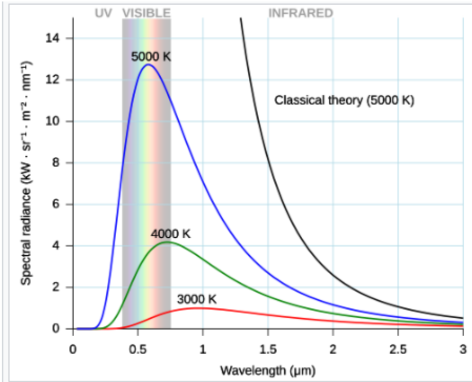
1a) The Planck model matches up well.

1b) I'm expecting something that mentions the break-down of the Rayliegh-Jeans prediction at short wavelengths.

Screen shot of what you'll find with Wikipedia:

The **ultraviolet catastrophe**, also called the **Rayleigh–Jeans catastrophe**, was the prediction of late 19th century and early 20th century classical physics that an ideal black body at thermal equilibrium would emit an unbounded quantity of energy as wavelength decreased into the ultraviolet range.[1]:6–7 The term "ultraviolet catastrophe" was first used in 1911 by Paul Ehrenfest,[2] but the concept originated with the 1900 statistical derivation of the Rayleigh–Jeans law.

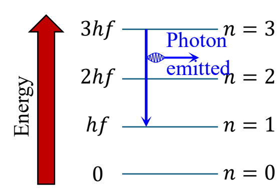
The phrase refers to the fact that the empirically derived Rayleigh–Jeans law, which accurately predicted experimental results at large wavelengths, failed to do so for short wavelengths. (See the image for further elaboration.) As the theory diverged from empirical observations when these frequencies reached the ultraviolet region of the electromagnetic spectrum, there was a problem.[3] This problem was later found to be due to a property of quanta as proposed by Max Planck: There could be no fraction of a discrete energy package already carrying minimal energy.



The ultraviolet catastrophe is the error at short wavelengths in the Rayleigh–Jeans law (depicted as "classical theory" in the graph) for the energy emitted by an ideal black body. The error, much more pronounced for short wavelengths, is the difference between the black curve (as classically predicted by the Rayleigh–Jeans law) and the blue curve (the measured curve, predicted by Planck's law).

1c) I'm expecting something about quantized energy levels.

From the workbook page 67:

Planck's Hypothesis	A black body is made up of atomic oscillators. 1) The energy of each oscillator can only take on certain discrete (quantized) energy levels. $E_n = n^{th}$ energy level $= nhf$ $h = 6.626 \times 10^{-34}\ J \cdot s$  2) The oscillators absorb/emit radiation during a transition from an initial state (n_i) to a final state (n_f). In the figure at right $n_i = 3, n_f = 1, \& \Delta E = 2hf$. Notice this transition to a lower energy causes a photon of energy $2hf$ to be emitted by the oscillator.

From an AI bot during a web search with the Brave browser:

Planck's Hypothesis

Planck's hypothesis, introduced in 1900, **suggested that the energy of oscillating electrons is not continuous but rather discrete, or quantized**. This means that energy is emitted and absorbed in discrete amounts, known as quanta, rather than in a continuous spectrum. 1 3 4 Planck proposed that the energy of a quantum of light (later termed a photon by Einstein) is given by the equation $E = nh\nu$, where h is Planck's constant (6.626×10^{-34} Js), and n is an integer (now called a quantum number). 2 4 This hypothesis resolved the ultraviolet catastrophe, a discrepancy between the classical theory of thermal radiation and experimental results, particularly at higher frequencies. 2 4

2a) Unless you require many sig figs, it is typically reasonable to ignore special relativity whenever the Lorentz factor (γ) is close to 1.00. In this problem $\beta = \frac{v}{c} = 0.0125$.

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1.000078$$

Relativity is not a concern.

2b) Kinetic energy of a photoelectron is given by

$$K_{max} = E_\gamma - \phi$$

$$\frac{1}{2}mv^2 = \frac{hc}{\lambda} - \phi$$

$$\frac{hc}{\lambda} = \frac{1}{2}mv^2 + \phi$$

$$\lambda = \frac{hc}{\frac{1}{2}mv^2 + \phi}$$

Think: it will be handy to use $hc = 124\underline{0}$ eV. Determine $\frac{1}{2}mv^2$ in units of eV. Use $m_e = 0.511\,\frac{\text{MeV}}{c^2}$.
Look up the work function of aluminum on the equation sheet as well.

$$\lambda = \frac{124\underline{0}\ \text{eV}}{\frac{1}{2}\left(0.511 \times 10^6\,\frac{\text{eV}}{c^2}\right)(0.0125c)^2 + 4.08\ \text{eV}}$$

$$\boldsymbol{\lambda = 28.2\ \text{nm}}$$

3a) Notice: on this question it is acceptable to answer with 3 sig figs (standard exam rules) or with the implied number of sig figs (which is actually 5 in this unusual case).
Did you remember the negative sign? Energy values of the hydrogen atom are negative.

$$E_6 = -\frac{13.606\ \text{eV}}{6^2} = -0.37794\underline{4}\ \text{eV}$$

3b) The atom absorbed a photon. Going to a higher n state implies going to a *less negative* state (energy increases).
3c) Determine E_8 the same way and use it to determine $|\Delta E| = E_\gamma$.

$$E_8 = -0.2125\underline{9}4\ \text{eV}$$
$$|\Delta E| = E_8 - E_6 = 0.1653\underline{5}1\ \text{eV}$$

The wavelength of light is given by

$$E_\gamma = \frac{hc}{\lambda} = |\Delta E|$$

The momentum of light is given by

$$p = \frac{h}{\lambda} = \frac{|\Delta E|}{c} = 0.1653\underline{5}1\,\frac{\text{eV}}{c} = \mathbf{8.836 \times 10^{-29}\ kg \cdot \frac{m}{s}}$$

Notice the result used the conversion 1.602×10^{-19} J = 1 eV.
You shouldn't be writing that last form with five sig figs.
Notice it may be wise on exams to stick to the exam rules and use three sig figs on final results.
Engineers have it so easy…right?

4a) **Shorter.** In Frame $\mathbf{S}'$, the rod is moving to the left. In a frame where the rod is moving, length contracts.

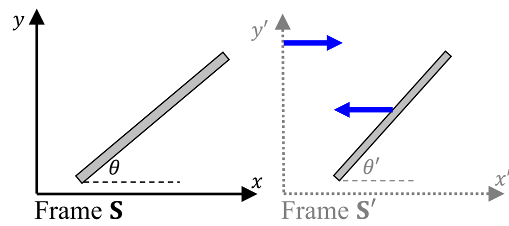
4b) Span in the x-direction *decreases* $(L'_x < L_x)$ but span in the y-direction *doesn't* change $(L'_y = L_y)$.

$$\boldsymbol{\theta < \theta'}$$

4c) In Frame $\mathbf{S}$, the horizontal & vertical spans are

$$L_x = (2.50\text{ cm})\cos 40.0° = 1.9\underline{1}51\text{ cm}$$

$$L_y = (2.50\text{ cm})\sin 40.0° = 1.6\underline{0}70\text{ cm}$$



The Lorentz factor in this problem is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(0.640)^2}} = 1.3\underline{0}14$$

The contracted horizontal span in Frame $\mathbf{S}'$ is

$$L'_x = \frac{L_x}{\gamma} = \frac{1.9\underline{1}5\text{ cm}}{1.3014} = 1.4\underline{7}16\text{ cm}$$

The new rod length is thus

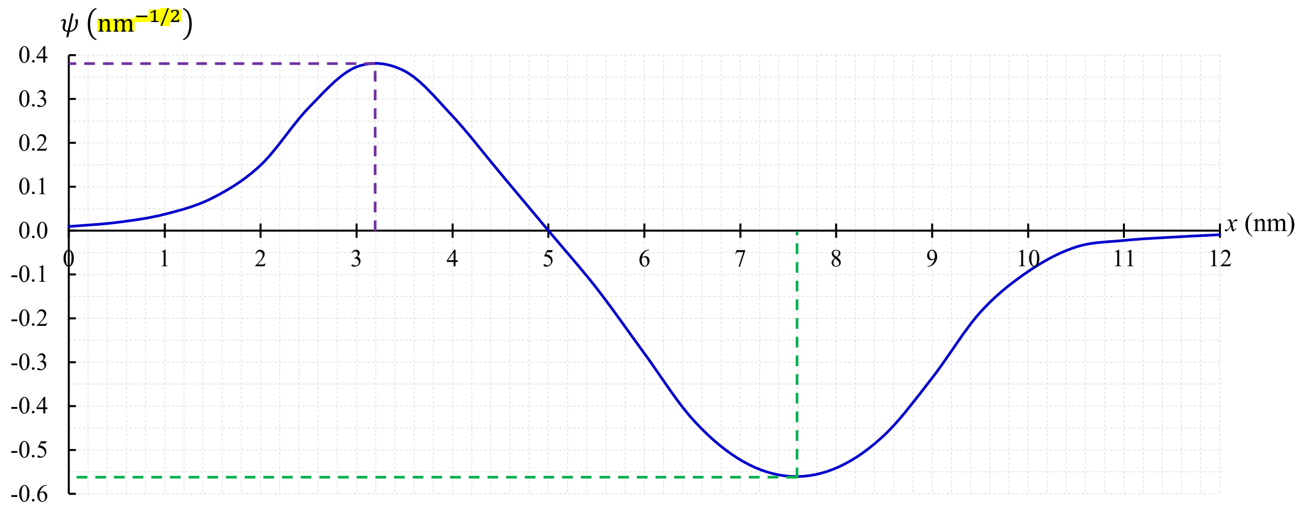
$$L = \sqrt{L'^2_x + L'^2_y}$$

WATCH OUT! The vertical span is unchanged!

$$L = \sqrt{L'^2_x + L^2_y}$$

$$L = \sqrt{(1.4\underline{7}16\text{ cm})^2 + (1.6\underline{0}70\text{ cm})^2}$$

$$\boldsymbol{L = 2.18\text{ cm}}$$



5a) **Answer:** nm$^{-1/2}$.
If you answered m$^{-1/2}$ I gave you half credit.
We were told this plot is *normalized*.
Total area under the square of the curve must be 1.
This happens here if units on the vertical axis equal $\frac{1}{\sqrt{\text{units on } x\text{-axis}}}$.

5b) **Answer:** At point $x = 5.00$ nm. I expect you to include units! I specifically told you to ignore the endpoints 0 & 12 nm.
The probability distribution (for a 1D problem) is given by

$$P(x) = \psi^*(x)\psi(x)$$

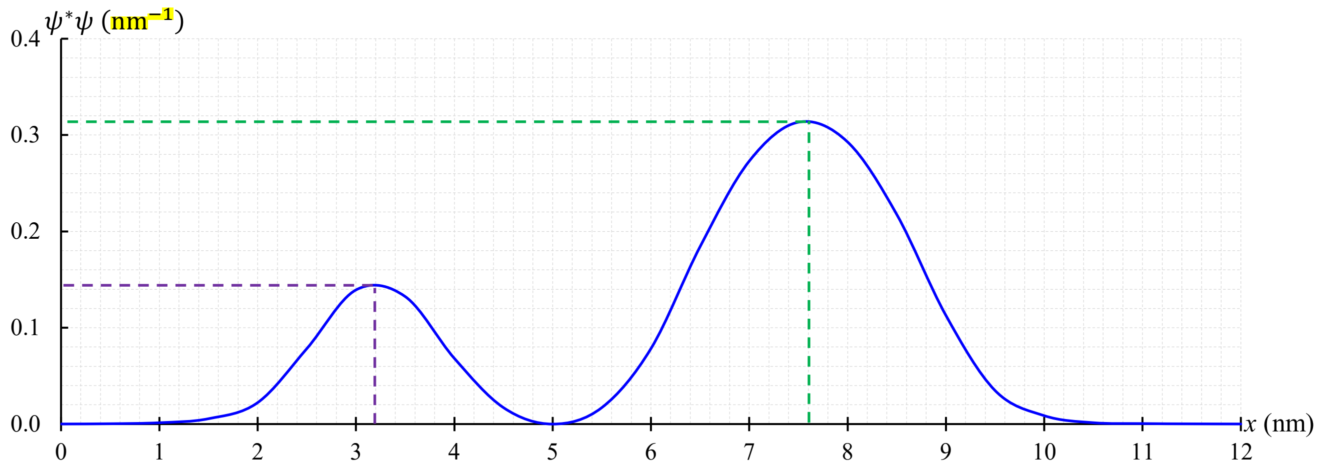
Zero probability occurs whenever $P(x) = 0$.
Notice this also happens when $\psi(x) = 0$

5c) **Answer:** at approximately $x = 7.6$ nm. In the plot below we can see the probability of being found at 3.2 nm is much less likely.
For a real-valued wavefunction

$$P(x) = \psi^*(x)\psi(x) = \psi^2(x)$$

The value of $\psi^2(7.6\text{ nm})$ is *significantly* larger than $\psi^2(x)$ for all other x.

5d) My plot is shown below. Notice the peak heights and locations. Notice the label with units on the vertical axis.



6a) **Answer:** $\psi(L) = 0$. The problem statement clearly indicates $U(L) = \infty$. We expect zero probability of the particle being found at $x = L$. This implies the wavefunction is zero at $x = L$.

6b) Best answer was "**None of the other answers is correct.**"
For the wavefunction to go to zero at both $x = 0$ & $x = L$ we assume

$$\psi(x) = \sqrt{\frac{2}{L}}\sin(kx)$$

where wavenumber k relates to wavelength using

$$k = \frac{2\pi}{\lambda}$$

The wavelengths which fit require n half-wavelengths fit in the length of the well.

$$L = n\left(\frac{\lambda}{2}\right) \quad \rightarrow \quad \lambda_n = \frac{2L}{n}$$

This gives

$$k = \frac{2\pi}{\frac{2L}{n}} = \boldsymbol{\frac{n\pi}{L}}$$

You could've also looked at the wavefunction for particle-in-a-box given on the equation sheet to see this…

6c) Expectation value of kinetic energy is given by

$$\langle\hat{K}\rangle = \langle\frac{\hat{p}^2}{2m}\rangle = \frac{1}{2m}\int_0^L \psi^*(x)\left(i\hbar\frac{\partial}{\partial x}\right)^2 \psi(x)\, dx$$

$$\langle\hat{K}\rangle = -\frac{\hbar^2}{2m}\int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{\pi n x}{L}\right)\frac{\partial^2}{\partial x^2}\sqrt{\frac{2}{L}}\sin\left(\frac{\pi n x}{L}\right) dx$$

$$\langle\hat{K}\rangle = -\frac{\hbar^2}{2m}\cdot\frac{2}{L}\int_0^L \sin\left(\frac{\pi n x}{L}\right)\frac{\partial^2}{\partial x^2}\sin\left(\frac{\pi n x}{L}\right) dx$$

$$\langle\hat{K}\rangle = -\frac{\hbar^2}{2m}\cdot\frac{2}{L}\int_0^L \sin\left(\frac{\pi n x}{L}\right)\left[-\left(\frac{\pi n}{L}\right)^2\right]\sin\left(\frac{\pi n x}{L}\right) dx$$

$$\langle\hat{K}\rangle = \frac{\hbar^2}{2m}\cdot\frac{2}{L}\cdot\left(\frac{\pi n}{L}\right)^2\int_0^L \sin^2\left(\frac{\pi n x}{L}\right) dx$$

$$\langle\hat{K}\rangle = \frac{\hbar^2}{2m}\cdot\frac{2}{L}\cdot\left(\frac{\pi n}{L}\right)^2\left[\frac{x}{2} - \frac{\sin\left(\frac{2\pi n x}{L}\right)}{4\frac{\pi n}{L}}\right]_0^L$$

$$\langle\hat{K}\rangle = \frac{\hbar^2}{2m}\cdot\frac{2}{L}\cdot\left(\frac{\pi n}{L}\right)^2\left[\left(\frac{L}{2} - 0\right) - (0 - 0)\right]$$

$$\langle\hat{K}\rangle = \frac{\hbar^2}{2m}\cdot\frac{2}{L}\cdot\left(\frac{\pi n}{L}\right)^2\cdot\frac{L}{2}$$

$$\boldsymbol{\langle\hat{K}\rangle = \frac{\hbar^2\pi^2 n^2}{2mL^2} \quad \text{or} \quad \frac{h^2 n^2}{8mL^2} \quad \text{where } n = 1, 2, 3, \ldots}$$

Note: to get the form on the right I plugged in $\hbar = \frac{h}{2\pi}$. Either of these forms was full credit on the exam.

7a) We are asked to determine the normalization constant for a 3D radial wavefunction. Don't forget the $4\pi r^2$! Also, notice the limits are given by the smallest possible radius ($r = 0$) to the largest ($r = \infty$).

$$1 = \int_0^\infty \psi^*(r)\, \psi(r)\, 4\pi r^2\, dr$$

For a real-valued wavefunction, $\psi^*(r) = \psi(r)$ and thus

$$\psi^*(r)\, \psi(r) = \psi^2(r) = \left(Ae^{-9r/2L}\right)\left(Ae^{-9r/2L}\right) = A^2\left(e^{-9r/2L}\right)^2 = A^2 e^{-9r/L} = A^2 e^{-ar^2} \quad \text{where } a = \frac{9}{L}$$

$$1 = \int_0^\infty A^2 e^{-ar}\, 4\pi r^2\, dr$$

$$1 = 4\pi A^2 \int_0^\infty e^{-ar}\, r^2\, dr$$

Use the integral table to find

$$1 = 4\pi A^2 \left[-\frac{e^{-ar}}{a^3}(a^2 r^2 + 2ar + 2)\right]_0^\infty$$

Think: from experience you should already know the $r = \infty$ drops out for all terms because $e^{-\infty} \to 0$.
Also, we can flip limits to get rid of the minus sign.

$$1 = 4\pi A^2 \left[+\frac{e^{-ar}}{a^3}(a^2 r^2 + 2ar + 2)\right]_\infty^0$$

Notice $e^{-0} = 1$. Also notice the first two terms in parentheses both drop out when $r = 0$!
The only surviving term upon plugging in the limits is

$$1 = 4\pi A^2 \cdot \left(\frac{2}{a^3}\right)$$

$$A^2 = \frac{a^3}{8\pi}$$

$$A = \sqrt{\frac{a^3}{8\pi}}$$

Now plug in $a = \frac{9}{L}$.

$$A = \sqrt{\frac{\left(\frac{9}{L}\right)^3}{8\pi}}$$

$$\boldsymbol{A = \sqrt{\frac{729}{8\pi L^3}}}$$

This final answer is simplified and in typical form for a physics class.
If you are an engineer, it is good practice to reduce all numerical factors into a single constant sitting out front of the radical like this:

$$\boldsymbol{A = \frac{5.39}{L^{3/2}}}$$

Either of these last two forms was full credit.