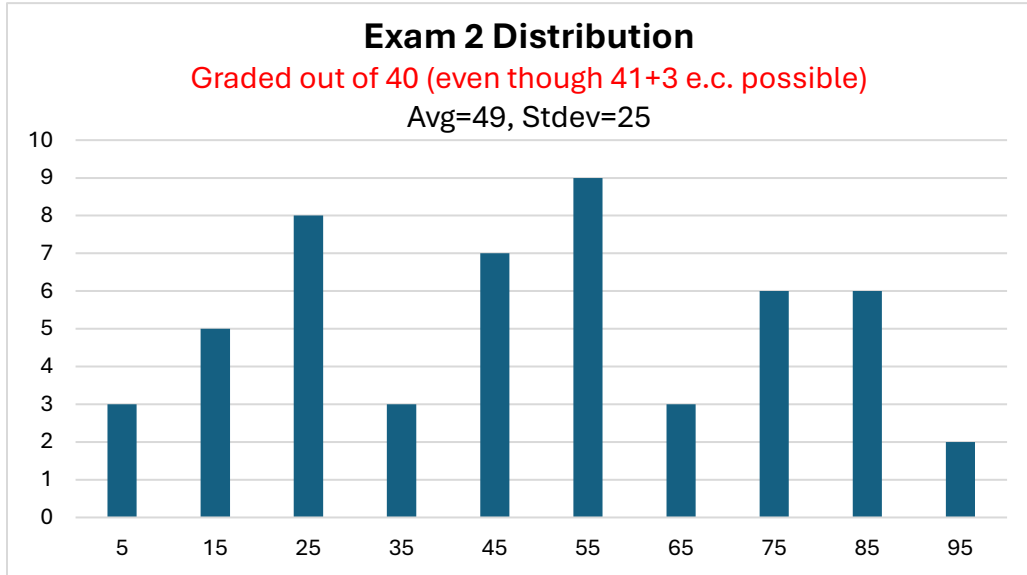


162sp26t2aSoln

Distribution on this page.

Solutions begin on the next page.



1) Follow the standard process.

- Determine moment of inertia about the pivot.
- Use sum of torques about the pivot to write a differential equation.
- Simplify the differential equation using the small angle approximation.
- Identify ω_0 in the differential equation and convert to T .

Notice we must use the parallel axis theorem since the pivot location doesn't match the table.

WATCH OUT! When using the parallel axis theorem, we must use $I_{cm\ rod}$ not $I_{end\ rod}$!

$$I = \frac{1}{12}mL^2 + m\left(\frac{L}{4}\right)^2 = \frac{7}{48}mL^2 \approx 0.14583mL^2$$

Now consider the figure at right.

The reaction forces at the pivot cause no torques (line of action through pivot).

I will assume \vec{a} is in the $+\hat{k}$ direction (but expect a negative value).

$$\Sigma \vec{\tau} = I\vec{a}$$

Notice both torques are exerted in the $-\hat{k}$ direction.

$$-r_{mg}F_{mg} \sin \theta_{mg} \hat{k} - r_{kx}F_{kx} \sin \theta_{kx} \hat{k} = I\ddot{\theta} \hat{k}$$

$$-\left(\frac{L}{4}\right)mg \sin \theta \hat{k} - \left(\frac{3}{4}L\right)k\left(\frac{3}{4}L \sin \theta\right) \sin(90^\circ - \theta) \hat{k} = \frac{7}{48}mL^2 \ddot{\theta} \hat{k}$$

Notice we can now cancel all the \hat{k} 's.

When using the small angle approximation, we expect $\sin \theta \approx \theta$ & $\cos \theta \approx 1$.

Notice we then use $\sin(90^\circ - \theta) = \cos \theta \approx 1$.

$$-\frac{1}{4}mgL\theta - \frac{9}{16}kL^2\theta(1) = \frac{7}{48}mL^2\ddot{\theta}$$

Solve for $\ddot{\theta}$ and simplify so it is easier to identify ω_0 . Notice what happens when you factor out the minus sign!

$$\ddot{\theta} = -\left(\frac{\frac{1}{4}mgL + \frac{9}{16}kL^2}{\frac{7}{48}mL^2}\right)\theta$$

$$\ddot{\theta} = -\left(\frac{12mg + 9kL}{7mL}\right)\theta$$

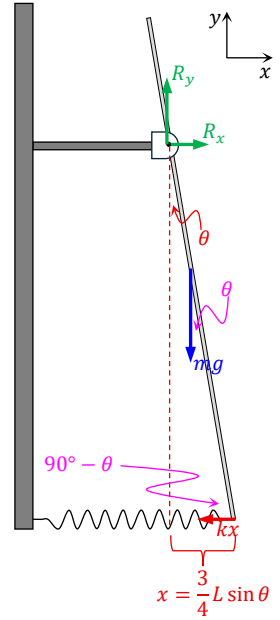
Identify ω_0 .

$$\omega_0 = \sqrt{\frac{12mg + 9kL}{7mL}}$$

Convert to period.

$$T = \frac{2\pi}{\omega_0}$$

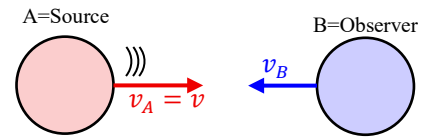
$$T = 2\pi \sqrt{\frac{7mL}{12mg + 9kL}}$$



2) A *driven* oscillator always oscillates at the angular frequency used by the driver. This is true whether the system has a large amount or a small amount of damping. A damped oscillator *without a driver* is different. Read notes in Volume 3 pages 19-20 for more.

3) Notice the subtle change in wording for this question. We are told there is damping and the student pulls the mass to the right and releases it from rest. This implies there is no driver for question 3! Critical damping allows the mass to reach final equilibrium most rapidly.

4a) Drawing a picture is probably useful. It helps get signs correct. Because the objects are coming together, the frequency is shifted up.



4b) Using the doppler shift formula gives

$$f' = f_0 \frac{c \pm v_{\text{observer}}}{c \mp v_{\text{source}}}$$

$$\begin{aligned} v_B &= 22.5\% \text{ less than } v_A \\ v_B &= (1 - 0.225)v \\ v_B &= 0.775v \end{aligned}$$

You could just read the equation sheet to get the signs correct.

Alternatively, think:

- Want numerator to get bigger for an upshift caused by observer moving towards source: use +.
- Want denominator to get smaller for an upshift caused by source moving towards observer: use -.

$$f' = f_0 \frac{c + 0.775v}{c - v}$$

Now think! We know the frequency is upshifted by 5.00%. This implies

$$f' = 5.00\% \text{ more than } f_0 = (1 + 5.00\%)f_0 = 1.0500f_0$$

Plug this in and solve for v .

$$1.0500f_0 = f_0 \frac{c + 0.775v}{c - v}$$

$$1.0500(c - v) = c + 0.775v$$

$$0.0500c = 1.0500v + 0.775v$$

$$0.0500c = 1.825v$$

$$v = \frac{0.0500}{1.825}c$$

$$v = 0.0274c$$

$$5a) k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.300 \text{ m}} = 20.94 \text{ m}^{-1}$$

$$5b) v = \frac{\lambda}{T} \rightarrow T = \frac{\lambda}{v} = \frac{0.300 \text{ m}}{15.00 \frac{\text{m}}{\text{s}}} = 20.00 \text{ ms}$$

$$5c) \omega = \frac{2\pi}{T} = 314.2 \frac{\text{rad}}{\text{s}}$$

5d) Start by using the equation sheet to derive a formula for $v_y(x, t)$. **Moving left implies $+\omega t$!**

$$v_y = \frac{\partial}{\partial t} y = \frac{\partial}{\partial t} A \sin(kx + \omega t + \phi) = +\omega A \cos(kx + \omega t + \phi)$$

We were asked for initial ($t = 0$) vertical velocity for the string segment at the origin ($x = 0$). Plug in these values.

$$v_y = \omega A \cos(0 + 0 + 90.0^\circ)$$

$$v_y = \omega A \cos(90.0^\circ)$$

$$v_y = 0$$

$$5e) v = \sqrt{\frac{\text{Tension}}{\mu}} \rightarrow \mu = \frac{\text{Tension}}{v^2} = \frac{5.55 \text{ N}}{(15.00 \frac{\text{m}}{\text{s}})^2} = 0.02467 \frac{\text{kg}}{\text{m}} = 24.7 \frac{\text{g}}{\text{m}}$$

5f) Since it is standard in some circles to express phase angles in degrees (even when the other parameters use radians) I will accept either of the following. Be sure to state units on all terms, not just the phase angle!

$$y(x, t) = (4.00 \text{ mm}) \sin \left[(20.94 \text{ m}^{-1})x + \left(314.2 \frac{\text{rad}}{\text{s}} \right) t + (90.0^\circ) \right]$$

$$y(x, t) = (4.00 \text{ mm}) \sin \left[(20.94 \text{ m}^{-1})x + \left(314.2 \frac{\text{rad}}{\text{s}} \right) t + \left(\frac{\pi}{2} \text{ rad} \right) \right]$$

5g) We now restrict ourselves to $x = 0$ because we care about the vertical position of the segment at the origin.

$$y(x, t) = (4.00 \text{ mm}) \sin \left[\left(314.2 \frac{\text{rad}}{\text{s}} \right) t + \left(\frac{\pi}{2} \text{ rad} \right) \right]$$

If you are clever you can first use $\sin(\theta + 90^\circ) = \cos \theta$

$$y(x, t) = (4.00 \text{ mm}) \cos \left[\left(314.2 \frac{\text{rad}}{\text{s}} \right) t \right]$$

Notice the function should be a simple cosine function with period 20.0 ms and amplitude 4.00 mm.

If you didn't see this, the easiest thing to do is tabulate y for $t = 0$ & $t = \frac{T}{4} = 5.00 \text{ ms}$.

Once you have the first two points the rest is easy (I hope).

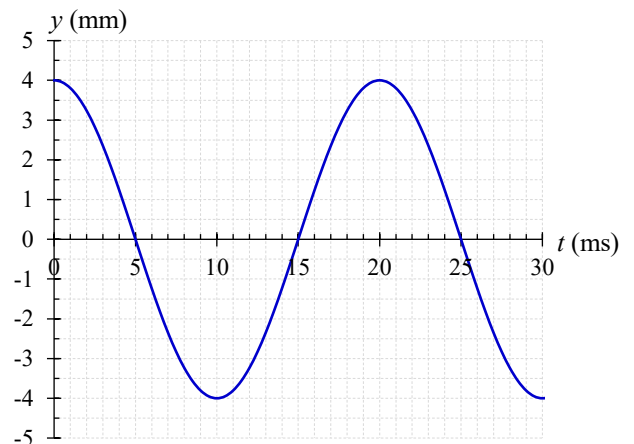
$$y(0, 0) = (4.00 \text{ mm}) \sin \left[0 + \left(\frac{\pi}{2} \right) \right] = 4.00 \text{ mm}$$

$$y(0, 5.00 \text{ ms}) = (4.00 \text{ mm}) \sin \left[\left(314.2 \frac{\text{rad}}{\text{s}} \right) (5.00 \times 10^{-3} \text{ s}) + \left(\frac{\pi}{2} \text{ rad} \right) \right]$$

$$y(0, 5.00 \text{ ms}) = 0$$

Double check the mode in your calculator before you plug in!

With everything in radians, use radians mode!



6a) We are asked about time of travel for a wave pulse on a string

When you see this kind of problem, in general you should think about the following:

$$v = \frac{dx}{dt} \rightarrow dt = \frac{dx}{v} \rightarrow \Delta t = \int_i^f \frac{dx}{v}$$

We know wave speed is given by

$$v = \sqrt{\frac{\text{Tension}}{\mu}} = \sqrt{\frac{T}{\mu}}$$

Therefore

$$\Delta t = \int_{\text{mass position}}^{\text{wall position}} \frac{dx}{v} = \int_{\text{mass position}}^{\text{wall position}} \sqrt{\frac{\mu}{T}} dx$$

Because the string mass is negligible compared to the hanging mass, we expect this tension is constant throughout the string. **Linear mass density** (μ) of the string is also constant (unless otherwise specified). Notice wave speed is constant.

$$\Delta t = \int_{\text{mass position}}^{\text{wall position}} \sqrt{\frac{\mu}{T}} dx \rightarrow \Delta t = \sqrt{\frac{\mu}{T}} \Delta x$$

Use a free body diagram to determine tension. Note: we typically assume a mass hanging on string is in equilibrium unless otherwise specified.

$$2T \cos \theta = Mg$$

$$T = \frac{Mg}{2 \cos \theta}$$

Now use the lowest figure at right to learn about distances.

$$\sin \theta = \frac{\frac{d}{2}}{\Delta x} \rightarrow \Delta x = \frac{d}{2 \sin \theta}$$

*Notice total string length is thus $L = 2\Delta x = \frac{d}{\sin \theta}$

**Notice this implies linear mass density is

$$\mu = \frac{m_{\text{string}}}{L} = \frac{m_{\text{string}} \sin \theta}{d}$$

Plug all this into the equation for Δt and solve for m_{string} .

$$\Delta t = \sqrt{\frac{\mu}{T}} \Delta x$$

$$\Delta t^2 = \frac{\mu}{T} \Delta x^2$$

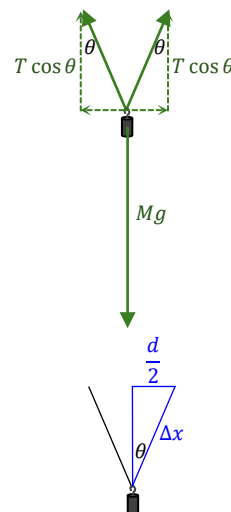
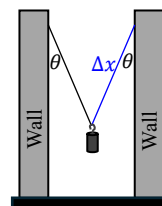
$$\Delta t^2 = \frac{\frac{m_{\text{string}} \sin \theta}{d}}{\frac{Mg}{2 \cos \theta}} \left(\frac{d}{2 \sin \theta} \right)^2$$

$$\Delta t^2 = \frac{m_{\text{string}} \sin \theta}{d} \cdot \frac{2 \cos \theta}{Mg} \cdot \frac{d^2}{4 \sin^2 \theta}$$

$$m_{\text{string}} = \frac{2g\Delta t^2 \tan \theta}{d} M$$

It is standard practice to answer questions in terms of given parameters. Anyone doing any work knows this.

6b) **Expect $\Delta t' < \Delta t$.** As angle increases, tension increases, wave speed increases, time of travel decreases.



7) Relate intensity to dB using

$$\beta = 10 \text{ dB} \log_{10} \frac{I}{I_0}$$

$$\frac{\beta}{10 \text{ dB}} = \log_{10} \frac{I}{I_0}$$

$$10^{\beta/10 \text{ dB}} = \frac{I}{I_0}$$

$$I_0 10^{\beta/10 \text{ dB}} = I$$

We were told power is radiated *hemispherically*. The power spreads out over *half* the area of a sphere.

$$I_0 10^{\beta/10 \text{ dB}} = \frac{\mathcal{P}}{2\pi r^2}$$

$$\mathcal{P} = 2\pi r^2 I_0 10^{\beta/10 \text{ dB}}$$

$$\mathcal{P} = 2\pi (0.888 \text{ m})^2 \left(10^{-12} \frac{\text{W}}{\text{m}^2} \right) 10^{122.5 \text{ dB}/10 \text{ dB}}$$

$$\mathcal{P} = \mathbf{8.81 \text{ W}}$$

8a) Don't overthink it.

$$v_{\text{sound air}} \approx 331.4 \frac{\text{m}}{\text{s}} + \left(0.61 \frac{\text{m}}{\text{s} \cdot ^\circ\text{C}}\right) T_C$$

$$v_{\text{sound air}} \approx 331.4 \frac{\text{m}}{\text{s}} + \left(0.61 \frac{\text{m}}{\text{s} \cdot ^\circ\text{C}}\right) (40.0 ^\circ\text{C})$$

$$v_{\text{sound air}} \approx \mathbf{355.8 \frac{\text{m}}{\text{s}}} \approx \mathbf{356 \frac{\text{m}}{\text{s}}}$$

On test days we typically round final results to three sig figs. If properly tracking sig figs, we could get four sig figs. For this one answer I'll accept 3 or 4 sig figs. That said, this is only an estimate, going to four sig figs is a bit silly.

8b) On multiple occasions we have derived

$$f_{\text{mixed}} = \frac{nv}{4L} \text{ where } n = 1, 3, 5, \dots \quad \& \quad f_{\text{matched}} = \frac{nv}{2L} \text{ where } n = 1, 2, 3, \dots$$

Consider the frequency difference between the lowest two values of n for each case

$$\Delta f_{\text{mixed}} = \left(\frac{3v}{4L}\right) - \left(\frac{v}{4L}\right) = \frac{v}{2L} \quad \& \quad \Delta f_{\text{matched}} = \left(\frac{2v}{2L}\right) - \left(\frac{v}{2L}\right) = \frac{v}{2L}$$

Notice the resonant frequencies are all separated by the same Δf .

In this problem, the two frequencies are separated by $\Delta f = 80 \text{ Hz}$.

Keep subtracting by 80 Hz until you can't anymore!

$$280 \text{ Hz} \rightarrow 200 \text{ Hz} \rightarrow 120 \text{ Hz} \rightarrow 40 \text{ Hz}$$

The fundamental frequency is 40 Hz . On exam days we use 3 sig figs but following proper sig fig rules gives 2 sig figs. For this part of problem I'll accept either 3 sig figs or 2 sig figs.

8c) Notice the fundamental of a matched state is $f_{1 \text{ matched}} = \Delta f$.

Notice the fundamental for a mixed state is $f_{1 \text{ mixed}} = \frac{1}{2} \Delta f$.

In our situation, $f_1 = \frac{1}{2} \Delta f$.

We have MIXED boundary conditions.

8d) Even if you miss part c, we could use our equation for Δf to determine L and still get this part correct.

$$\Delta f = \frac{v}{2L} \rightarrow L = \frac{v}{2\Delta f} \rightarrow L = \frac{355.8 \frac{\text{m}}{\text{s}}}{2(80 \text{ Hz})} \rightarrow L = 2.22 \text{ m}$$

Again, for this part of the problem I'll accept either 3 sig figs or 2 sig figs.

9a) Notice the plot is a function of time. We can read period off the plot!
I found it easiest to use the points intersecting the horizontal axis.

Notice these points in time are separated by $\frac{T}{2}$.

$$T = 2(340 \mu\text{s} - 140 \mu\text{s}) = \mathbf{400 \mu\text{s}}$$

9b) Angular frequency $\omega_0 = \frac{2\pi}{T} = \mathbf{15708 \frac{\text{rad}}{\text{s}}} = \mathbf{15.71 \frac{\text{krad}}{\text{s}}}$.

Do NOT leave prefixes in the basement (e.g $0.01571 \frac{\text{rad}}{\mu\text{s}}$)! Simplify or lose points.

9c) Use

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$k = m\omega_0^2$$

$$k = (3.20 \times 10^{-10} \text{ kg}) \left(15708 \frac{\text{rad}}{\text{s}}\right)^2$$

$$k = \mathbf{0.07896 \frac{\text{N}}{\text{m}}} = \mathbf{79.0 \frac{\text{mN}}{\text{m}}}$$

Tip: when the units form a fraction and you want to use a prefix, usually you should put the prefix on the unit in the numerator.

9d) WATCH OUT! We were given a plot of velocity versus time. We can read off the velocity amplitude v_{max} !

$$v_{max} = \omega_0 x_{max}$$

$$x_{max} = \frac{v_{max}}{\omega_0}$$

$$x_{max} = \frac{26.0 \frac{\text{m}}{\text{s}}}{15708 \frac{\text{rad}}{\text{s}}}$$

$$x_{max} = 0.0016552 \text{ m}$$

$$x_{max} = \mathbf{1.655 \text{ mm}}$$

9e) Max force is given by

$$F_{max} = ma_{max} = m(\omega_0^2 x_{max}) = 0.00013069 \text{ N} = \mathbf{130.7 \mu\text{N}}$$

9f) Most of you probably determined $v_i \approx 21.0 \frac{\text{m}}{\text{s}}$ from the plot and did this:

$$v_i = -\omega_0 x_{max} \sin \phi \rightarrow \phi = \sin^{-1} \left(-\frac{v_i}{\omega_0 x_{max}} \right) = \sin^{-1} \left[-\frac{21.0 \frac{\text{m}}{\text{s}}}{(15708 \frac{\text{rad}}{\text{s}})(0.0016552 \text{ m})} \right] = \begin{matrix} -53.87^\circ & -0.9042 \text{ rad} \\ \text{or} & \text{or} \\ -126.13^\circ & -2.2014 \text{ rad} \end{matrix}$$

Notice this is exactly the same thing as saying

$$v_i = -v_{max} \sin \phi \rightarrow \phi = \sin^{-1} \left(-\frac{v_i}{v_{max}} \right) = \sin^{-1} \left[-\frac{21.0 \frac{\text{m}}{\text{s}}}{26.0 \frac{\text{m}}{\text{s}}} \right] = \begin{matrix} -53.87^\circ & -0.9042 \text{ rad} \\ \text{or} & \text{or} \\ -126.13^\circ & -2.2014 \text{ rad} \end{matrix}$$

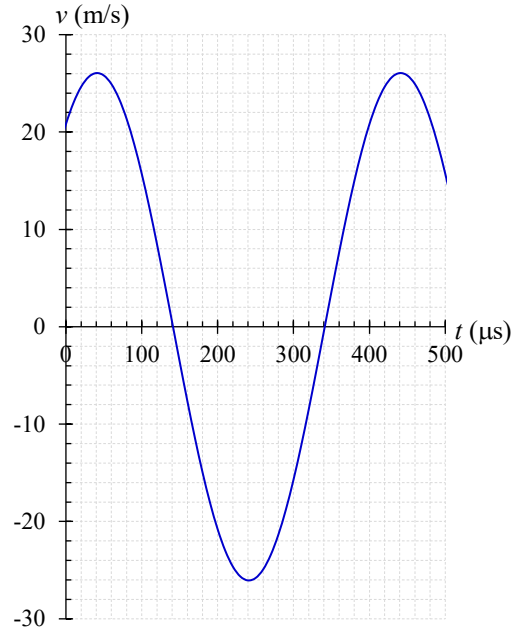
Now any other (non-zero) point on the curve to check this. I'm going to use $v = +26.0$ at time $t = 40 \mu\text{s}$.

WATCH OUT!!! I ended up using radians mode on my calculator because $\omega_0 t$ is in units of radians!!!

$$v_{@ t=40 \mu\text{s}} \stackrel{?}{=} -v_{max} \sin \left[\left(15708 \frac{\text{rad}}{\text{s}}\right) (40 \times 10^{-6}) + (-0.9042 \text{ rad}) \right] = 7.98 \frac{\text{m}}{\text{s}} = \text{incorrect!} \neq 26.0 \frac{\text{m}}{\text{s}}$$

$$v_{@ t=40 \mu\text{s}} \stackrel{?}{=} -v_{max} \sin \left[\left(15708 \frac{\text{rad}}{\text{s}}\right) (40 \times 10^{-6}) + (-2.2014 \text{ rad}) \right] = 26.0 \frac{\text{m}}{\text{s}} = \text{correct!}$$

Phase angle is thus $-2.20 \text{ rad} = -126.1^\circ$. I'll accept either answer.



If you are taking this test as a practice test, please actually do this problem.
It is good practice for future exams.

Extra Credit: Consider the figure shown at right.

We know there is no path length difference along the midline.

For speakers in phase, we expect *constructive* interference at all points along the midline.

Notice I indicated a *maximum* at the point where the midline intersects the horizontal axis.

We also know the origin and point **P** are *minima*.

I filled in the pattern of minima and maxima from there (told there were three maxima).

The first min (adjacent to the midline) corresponds to path length difference $\Delta r = \frac{\lambda}{2}$.

The first max off the midline corresponds to $\Delta r = \lambda = 2\left(\frac{\lambda}{2}\right)$.

The second min from the midline corresponds to $\Delta r = 3\left(\frac{\lambda}{2}\right) = \frac{3}{2}\lambda$.

Using some geometry one finds path length difference is

$$\Delta r_{origin} = r_2 - r_1$$

$$\frac{3}{2}\lambda = \sqrt{L^2 + d^2} - L$$

$$\sqrt{L^2 + d^2} = \frac{3}{2}\lambda + L$$

$$L^2 + d^2 = \left(\frac{3}{2}\lambda + L\right)^2$$

$$L^2 + d^2 = \frac{9}{4}\lambda^2 + 3\lambda L + L^2$$

$$d^2 = \frac{9}{4}\lambda^2 + 3\lambda L$$

$$d = \sqrt{3\lambda\left(L + \frac{3}{4}\lambda\right)}$$

$$d = \sqrt{\frac{3c}{f}\left(L + \frac{3c}{4f}\right)}$$

