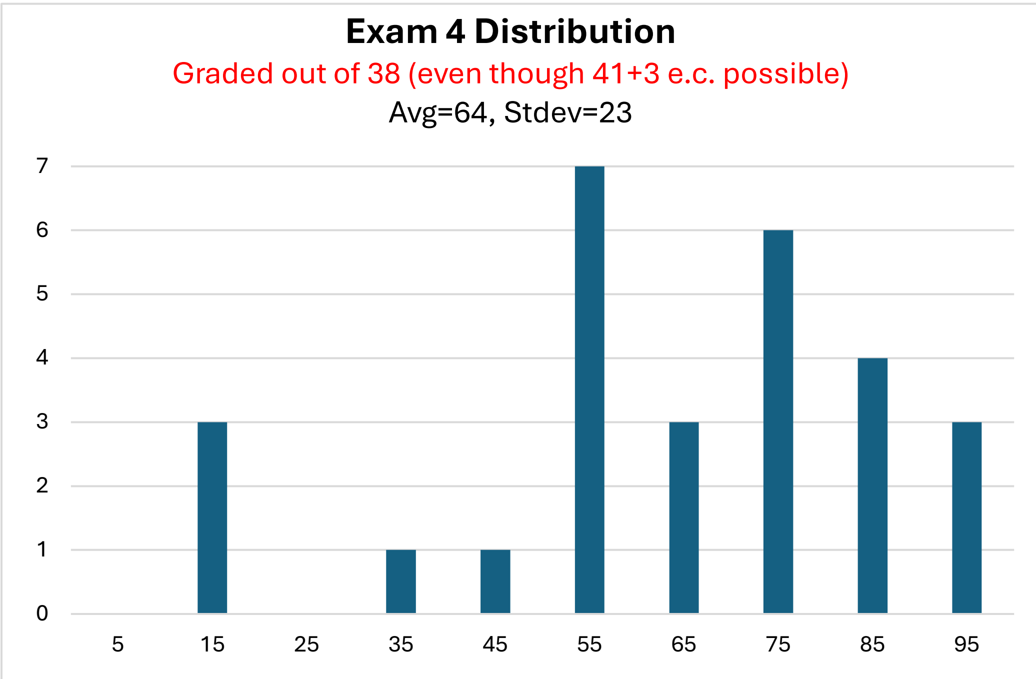
162sp26t4aSoln
Distribution on this page.
Solution begins on the next page.



1a) Strictly speaking $\beta = \frac{speed}{c}$ but we typically call β speed if we assume units of c.

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \rightarrow \quad \gamma^2 = \frac{1}{1-\beta^2} \quad \rightarrow \quad 1-\beta^2 = \frac{1}{\gamma^2} \quad \rightarrow \quad \beta^2 = 1 - \frac{1}{\gamma^2} \quad \rightarrow \quad \beta = \sqrt{1-\frac{1}{\gamma^2}} = \mathbf{0.94\underline{2}8}$$

1b)

$$K = (\gamma - 1)m_e c^2 = 2.00\left(0.511\frac{\text{MeV}}{c^2}\right)c^2 = \mathbf{1.022\ MeV}$$

1c) Use the unrounded result for β to avoid intermediate rounding error.

$$p = \|\vec{p}\| = \gamma m\|\vec{u}\| = 3.00\left(0.511\frac{\text{MeV}}{c^2}\right)(0.94\underline{2}8c) = \mathbf{1.445\frac{MeV}{c}}$$

2) *Long* wavelengths correspond to *low* energy photons (small energy changes).
Short wavelengths correspond to *high* energy photons (large energy changes).
To *emit* a photon, the hydrogen atom must go to a *lower* energy state.
Combining this information implies the hydrogen atom goes to $n = 1$ state to emit the shortest wavelength.

$$\Delta E = E_f - E_i$$

$$\Delta E = -13.606\text{ eV}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$$\Delta E = -13.606\text{ eV}\left(\frac{1}{1^2} - \frac{1}{11^2}\right)$$

$$\Delta E = -13.49\underline{3}6\text{ eV}$$

Emitted photon wavelength is given by

$$\frac{hc}{\lambda} = |\Delta E|$$

$$\lambda = \frac{hc}{|\Delta E|}$$

$$\lambda = \frac{124\underline{0}\text{ eV}\cdot\text{nm}}{13.49\underline{3}6\text{ eV}}$$

$$\boldsymbol{\lambda} = \mathbf{91.8\underline{9}5\ nm}$$

If you follow sig fig rules you actually get four sig figs on this number.
Note: because $124\underline{0}$ eV · nm starts with a 1, it acts more like a 3 sig fig number when used in computations (from a % error standpoint).

On exam days we typically round final answers to three sig figs.

$$\boldsymbol{\lambda} = \mathbf{91.9\ nm}$$

I accepted either 3 sig figs or 4 sig figs on your final result.

3a) The appropriate label for the vertical axis should be

$$\boldsymbol{\psi}\ (\mathbf{\mu m^{-1/2}})$$

Notice the units will cancel the units of the horizontal axis AFTER SQUARING!

3b) The probability distribution for a 1D particle is given by $\psi^*\psi$.
When ψ is real valued we may instead write $\psi^*\psi = \psi^2$.
Because this function drops to zero whenever $\psi = 0$, the question is essentially asking when is $\psi = 0$.
This happens at points $x = 2.00$ μm and $x = 5.00$ μm.
Note: since you read values from a graph, I'd accept 2 sig figs (realistic sig figs) or 3 sig figs (exam default).

3c) To determine the probability for a 1D problem, first compute ψ^2.
I made a quick plot of this shown at right.

$$Probability_{between\ x=2\ \&\ 5\ \mu m} = \int_{2.00\ \mu m}^{5.00\ \mu m} \psi^2\, dx$$

$$Probability_{between\ x=2\ \&\ 5\ \mu m} = Area\ under\ curve$$

$$Probability_{between\ x=2\ \&\ 5\ \mu m} = base \times height$$

Notice the value of $\psi^2 \approx 0.090\ \mu m^{-1}$ between $x = 2.00\ \&\ 5.00$ μm.

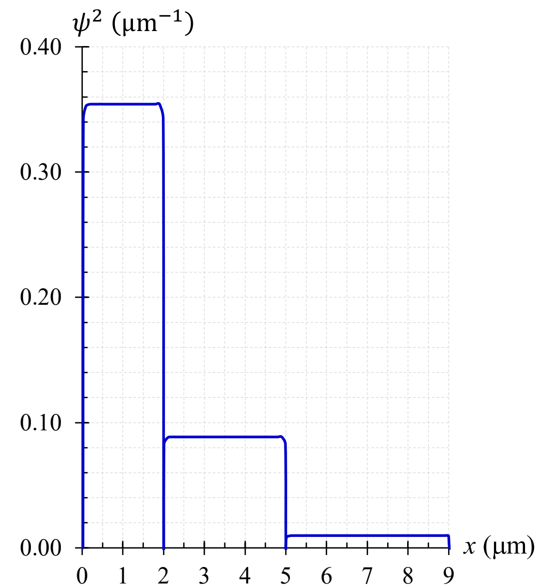
$$Probability_{between\ x=2\ \&\ 5\ \mu m} \approx (5.00\ \mu m - 2.00\ \mu m) \times (0.090\ \mu m^{-1})$$

$$Probability_{between\ x=2\ \&\ 5\ \mu m} = (3.00\ \mu m) \times (0.090\ \mu m^{-1})$$

$$Probability_{between\ x=2\ \&\ 5\ \mu m} = 0.27$$

$$\boldsymbol{Probability_{between\ x=2\ \&\ 5\ \mu m} = 27\%}$$

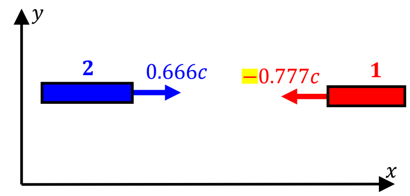
Again, since you read values from a graph, I'd accept 2 sig figs (realistic sig figs) or 3 sig figs (exam default).



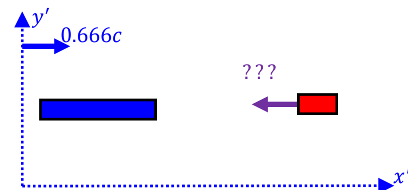
4a) Compton scattering is the elastic scattering of a photon off an electron.
When modelling elastic collisions, both conservation of momentum and conservation of energy are used.
WATCH OUT! one should use relativistic expressions for momentum and energy when using these laws!

4b) The photon always loses energy during Compton scattering.
The resulting shifted wavelength of the scattered photon is always *longer* (*lower* energy).

5a) The first thing I did was draw pictures to get a feeling for the situation. Since both ships have equal proper length, the one moving faster (relative to the rest frame) experiences more length contraction.
We expect $L_1 < L_2$ (as viewed by an observer at rest).



Rest Frame Perspective



Ship 1 Perspective

5b) The speed of ship 2 cannot exceed the speed of light; the answer is NOT 1.443c. Instead, we expect the relativistic velocity transformation should show

$$|v_{1\,rel\,2}| < c$$

5c)
Use the relativistic velocity transformation equation.

$$u_x = \frac{u'_x + v}{1 + \frac{v u'_x}{c^2}}$$

- $v = 0.666c$ is the velocity of the frame moving with ship 2
- $u_x = -0.777c$ is the velocity of ship 1 in the rest frame (check that minus sign!)
- u'_x is the velocity of ship 1 in the moving frame (what we want to know)

First rearrange the equation to solve for u'_x. Tip: check the units every few steps.

$$u_x\left(1 + \frac{v u'_x}{c^2}\right) = u'_x + v$$

$$u_x + \frac{v u'_x}{c^2} u_x = u'_x + v$$

$$u_x - v = u'_x - \frac{v u'_x}{c^2} u_x$$

$$u_x - v = u'_x\left(1 - \frac{v u_x}{c^2}\right)$$

$$u'_x = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

$$u'_x = \frac{-0.777c - 0.666c}{1 - \frac{(0.666c)(-0.777c)}{c^2}}$$

$$u'_x = \frac{-1.443c}{1 - (-0.5175)}$$

$$\boldsymbol{u'_x = -0.95\underline{0}9c}$$

$$\boldsymbol{u'_x = -2.85 \times 10^8 \frac{m}{s}}$$

I accepted either of the last two forms.

6) To determine a normalization constant for a 1D wavefunction use

$$1 = \int_{-\infty}^{+\infty} \psi^* \psi \, dx$$

$$1 = \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{3}}\psi_3^* + \alpha\psi_4^*\right)\left(\frac{1}{\sqrt{3}}\psi_3 + \alpha\psi_4\right) dx$$

$$1 = \int_{-\infty}^{+\infty} \left[\frac{1}{3}\psi_3^*\psi_3 + \frac{\alpha}{\sqrt{3}}(\psi_3^*\psi_4 + \psi_4^*\psi_3) + \alpha^2\psi_4^*\psi_4\right] dx$$

Hopefully you recall from the homework the following interesting result for stationary states (it's on the eq'n sheet):

$$0 = \int_{-\infty}^{+\infty} \psi_m^*(x)\psi_n(x) \, dx \text{ if } m \neq n$$

I showed this result graphically in the solution to problem **39.11** of the workbook.
This result eliminates the integrals of the two cross-terms!

$$1 = \int_{-\infty}^{+\infty} \left(\frac{1}{3}\psi_3^*\psi_3 + \alpha^2\psi_4^*\psi_4\right) dx$$

$$1 = \frac{1}{3}\int_{-\infty}^{+\infty} \psi_3^*\psi_3 \, dx + \alpha^2 \int_{-\infty}^{+\infty} \psi_4^*\psi_4 \, dx$$

By definition, $\int_{-\infty}^{+\infty} \psi_n^*\psi_n \, dx = 1$ for the two remaining integrals (*after* constants are factored out).

$$1 = \frac{1}{3} + \alpha^2$$

$$\boldsymbol{\alpha = \sqrt{\frac{2}{3}} \approx 0.816}$$

Note: if you didn't see this solution, you could use the wavefunctions given by

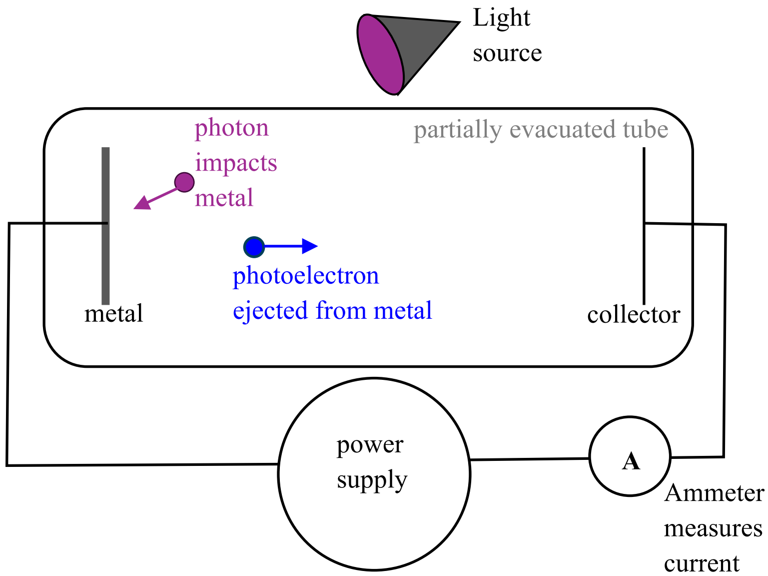
$$\psi_n = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi n x}{L}\right) \quad \textit{inside} \text{ the well and } \textit{zero} \text{ outside the well}$$

The integral becomes

$$1 = \int_{0}^{L} \psi^* \psi \, dx$$

You could work out the actual integrals using the trig function.

7a)



7b) In the photoelectric effect, only high energy photons (with frequencies *above* the cut-off frequency of the metal) can produce photoelectrons.

7c) Since we know photoelectrons are produced, we know source frequency exceeds cut-off frequency.
Increasing intensity produces more photoelectrons and thus more photocurrent.

7d) The maximum speed of photoelectrons is determined by frequency of light, not intensity.
Increasing source intensity produces more photoelectrons but has no effect on max speed!

7e) Stopping potential is the applied voltage required to prevent photoelectrons from reaching the collector.
One adjusts the power supply until the ammeter measures no current.
The absolute value of this voltage is called the stopping potential.

7f) Before the photoelectric effect was understood, all optical phenomena could be explained using the wave theory of light. The results of the photoelectric effect experiment could NOT be explained by assuming light was a *wave* but could be explained by assuming light was a *particle*.
This experiment forced the scientific community to accept the concept of particle-wave duality (in some experiments we must model light as a wave while in others we must model it as a particle)!

8) Classic Heisenberg Uncertainty Principle question.

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

$$\Delta p_x \geq \frac{\hbar}{2\Delta x}$$

$$m\Delta v_x \geq \frac{\hbar}{2\Delta x}$$

$$\Delta v_x \geq \frac{\hbar}{2m\Delta x}$$

$$\Delta v_x \geq \frac{1.0546 \times 10^{-34}\ \text{J·s}}{2(40.1 \times 1.673 \times 10^{-27}\ \text{kg})(3.00 \times 10^{-9}\ \text{m})}$$

$$\Delta v_x \geq 0.262\frac{\text{m}}{\text{s}}$$

Think: you might use a larger trap to have *even less* uncertainty if you are trying to make a quantum computer out of trapped atoms. Check out this link on trapped ion quantum computers…

9) For a 1D problem (for a small value of dx):

$$\begin{matrix} \textit{Probability of observation} \\ \textit{between } x \textit{ and } x + dx \end{matrix} = P(x)\,dx = \psi^*\psi\,dx$$

The radial probability density is derived from a 3D integral.

$$\begin{matrix} \textit{Probability of observation} \\ \textit{between } r \textit{ and } r + dr \end{matrix} = \psi^*\psi\,4\pi r^2\,dr$$

Said another way

$$P(a < r < b) = \int_{r=a}^{r=b}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \psi^*\psi\,r^2\sin\theta\,dr\,d\theta\,d\phi = \int_{r=a}^{r=b}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \psi^*\psi\,4\pi r^2 dr$$

Note: this result is valid because the wavefunction is spherically symmetric (no θ or ϕ dependence).
A similar process could be followed even if the wavefunction was not spherically symmetric.
That said, I wouldn't expect to see $\psi^*\psi\,4\pi r^2$ pop out, it would be something totally different.

Notice, with $\psi^*\psi\,4\pi r^2$ in the probability distribution we do get zero probability for radius $r = 0$.
To see the plot, visit the following website and scroll down to "Hydrogen 3s Radial Probability".
http://www.hyperphysics.phy-astr.gsu.edu/hbase/hydwf.html

10) To determine an energy value, plug the wave function into the Schrödinger equation and solve for E.

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+U\right)\psi = E\psi$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right)Bxe^{-kx^2} = EBxe^{-kx^2}$$

Notice the normalization constant B cancels immediately.

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right)xe^{-kx^2} = Exe^{-kx^2}$$

I'm going to do the derivatives first as a quick side problem.

$$\frac{d}{dx}xe^{-kx^2} = \frac{d}{dx}\left[e^{-kx^2}+x(-2kx)\,e^{-kx^2}\right] = e^{-kx^2}-2kx^2e^{-kx^2}$$

$$\frac{d^2}{dx^2}xe^{-kx^2} = \frac{d}{dx}\left(e^{-kx^2}-2kx^2e^{-kx^2}\right) = -2kxe^{-kx^2}-4kxe^{-kx^2}-2kx^2(-2kx)e^{-kx^2} = 4k^2x^3e^{-kx^2}-6kxe^{-kx^2}$$

Now plug this into the Schrödinger equation and solve for E.

$$-\frac{\hbar^2}{2m}\left(4k^2x^3e^{-kx^2}-6kxe^{-kx^2}\right)+\frac{1}{2}m\omega^2x^2\left(xe^{-kx^2}\right) = Exe^{-kx^2}$$

Notice every term has xe^{-kx^2} in it. Let's cancel that out now to make it easier clean things up.

$$-\frac{\hbar^2}{2m}\left(4k^2x^2-6k\right)+\frac{1}{2}m\omega^2x^2 = E$$

Since we have a stationary state, this result must hold for all values of x.
This can only be true if any x dependence on the left hand side drops out.
The trick here is to regroup the terms with x^2 in them.

$$-\frac{\hbar^2}{2m}\left(4k^2x^2\right)-\frac{\hbar^2}{2m}(-6k)+\frac{1}{2}m\omega^2x^2 = E$$

$$\frac{3\hbar^2}{m}k+\left(\frac{1}{2}m\omega^2-\frac{2\hbar^2k^2}{m}\right)x^2 = E$$

Must be zero else energy value would depend on x. Not true for a stationary state! Energy should be a constant!

Since the final result must have no x dependence, the term in parentheses must be zero!

$$\frac{1}{2}m\omega^2-\frac{2\hbar^2k^2}{m} = 0$$

$$\boldsymbol{k = \frac{m\omega}{2\hbar}}$$

Notice, now that the term in parentheses drops out, one finds

$$\frac{3\hbar^2}{m}k = E$$

Plug in the determined value of k.

$$\frac{3\hbar^2}{m}\left(\frac{m\omega}{2\hbar}\right) = E$$

$$\boldsymbol{E = \frac{3}{2}\hbar\omega}$$

Notice this is just the $n = 1$ (first excited state) of the quantum harmonic oscillator.

Extra Credit: To normalize the 1D wavefunction we use

$$1 = \int_{-\infty}^{+\infty} \psi^* \psi \, dx$$

Because this function is valid from $x = -\infty \rightarrow +\infty$, we see it is appropriate to keep the limits as is.
Because the wavefunction is *real*, we can use $\psi^* = \psi$. This reduces the integral to

$$1 = \int_{-\infty}^{+\infty} \psi^2 \, dx$$

$$1 = \int_{-\infty}^{+\infty} \left(Bxe^{-kx^2}\right)^2 dx$$

$$1 = B^2 \int_{-\infty}^{+\infty} x^2 e^{-2kx^2} \, dx$$

$$\frac{1}{B^2} = \int_{-\infty}^{+\infty} x^2 e^{-2kx^2} \, dx$$

Consult the tables given on the exam equation sheet.

$$\frac{1}{B^2} = \frac{1}{2} \cdot \sqrt{\frac{\pi}{(2k)^3}}$$

$$\frac{1}{B^2} = \sqrt{\frac{\pi}{32k^3}}$$

$$B = \left(\frac{32k^3}{\pi}\right)^{1/4}$$

$$\mathbf{B = 1.786\ k^{3/4}}$$

If you take the effort, you will see this does match the normalization constant for the first excited state of the quantum harmonic oscillator…