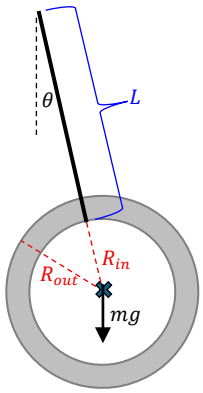
Ancient_t1_01_Soln

1) This looks like a period of oscillations problem.

$$\Sigma\tau = I\alpha$$



The only force causing a torque is gravity acting at the center of mass.
I will assume rotation CCW is the positive direction (out of the page).
Notice the gravitational torque is negative.
Also, I'm going to replace α with $\ddot{\theta}$.

$$-r_{cm}m_{total}g\sin(angle) = I\ddot{\theta}$$

$$-(L + R_{in})Mg\sin\theta = I_{\parallel}\ddot{\theta}$$

WATCH OUT! Use the parallel axis theorem on the tire!

$$-(L + R_{in})Mg\sin\theta = \left[\frac{1}{2}M(R_{in}^2 + R_{out}^2) + M(L + R_{in})^2\right]\ddot{\theta}$$

Now use the small angle approximation $\sin\theta = \theta$.

$$-\frac{2(L + R_{in})g}{R_{in}^2 + R_{out}^2 + 2(L + R_{in})^2}\theta = \ddot{\theta}$$

Identify ω_0 from the differential equation.

$$\omega_0 = \sqrt{\frac{2(L + R_{in})g}{R_{in}^2 + R_{out}^2 + 2(L + R_{in})^2}}$$

Convert to period.

$$\mathbb{T} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{R_{in}^2 + R_{out}^2 + 2(L + R_{in})^2}{2(L + R_{in})g}}$$

Now I will square both sides and move things around until both sides had units of meters (so I can ignore units).

$$\frac{g\mathbb{T}^2}{4\pi^2} = \frac{\frac{R_{in}^2 + R_{out}^2}{2} + (L + R_{in})^2}{L + R_{in}}$$

Plug in numbers and get a quadratic equation for L. Again, I already checked the units…

$$0.9929 = \frac{0.1250 + L^2 + 0.6000L + 0.09000}{L + 0.3000}$$

$$0.9929L + 0.2979 = 0.1250 + L^2 + 0.6000L + 0.09000$$

$$0 = L^2 - 0.3929L - 0.08288$$

Form there I did the quadratic formula and found two roots:

$$L = 0.545\text{ m} \quad \text{or} \quad -0.1521\text{ m}$$

In this scenario, a negative length doesn't make sense so we assume the positive root is correct.
Let's check it:

$$r_{cm} = L + R_{in} = 0.845\text{ m}$$

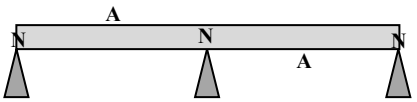
$$I_{\parallel} = \frac{1}{2}M(R_{in}^2 + R_{out}^2) + M(L + R_{in})^2 = 0.125M + 0.714M = (0.839\text{ m}^2)M$$

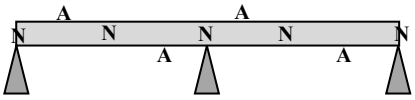
$$\mathbb{T} = 2\pi\sqrt{\frac{I}{Mgr_{cm}}} = \mathbb{T} = 2\pi\sqrt{\frac{(0.839\text{ m}^2)M}{M\left(9.8\frac{\text{m}}{\text{s}^2}\right)(0.845\text{ m})}} = 2.00\text{ s}$$

2a) Lower harmonics are lower frequencies.

2b) The 5th harmonic is a higher frequency than the 3rd harmonic. Higher frequencies have shorter wavelengths.

2c) The support at the middle restricts that position to a displacement *node*.
This constrains the relationship between bridge length and wavelength.
Draw the first two patterns of nodes & antinodes to get a feeling for the pattern.
Here I typically use one of the following facts:





- Distance between adjacent nodes is $\frac{\lambda}{2}$
- Distance between adjacent antinodes is $\frac{\lambda}{2}$
- Distance between a node and an adjacent antinode is $\frac{\lambda}{4}$

Notice the lowest achievable harmonic is $L = 2\left(\frac{\lambda}{2}\right) = \lambda$.

The next lowest achievable harmonic is $L = 4\left(\frac{\lambda}{2}\right) = 2\lambda$

In general one finds

$$L = n\lambda \text{ for } n = 1, 2, 3, \ldots \quad \rightarrow \quad \lambda = \frac{L}{n}$$

Now derive the frequencies as usual.

$$v = f\lambda$$

$$f = \frac{v}{\lambda}$$

$$f_n = \frac{nv}{L} \quad \text{for } n = 1, 2, 3, \ldots$$

3a) We are told linear mass density varies linearly from bottom to top. We expect

$$\mu(y) = slope \cdot y + intercept$$

Notice we have two unknowns ($slope$ & $intercept$).
Use the 2 known points to get 2 equations for the 2 unknowns.

At the bottom ($y = 0$)	**At the top ($y = 2.00$ m)**
$6.00\frac{\text{kg}}{\text{m}} = slope \cdot (0) + intercept$	$2.00\frac{\text{kg}}{\text{m}} = slope \cdot (2.00\text{ m}) + intercept$
$6.00\frac{\text{kg}}{\text{m}} = intercept$	$2.00\frac{\text{kg}}{\text{m}} = slope \cdot (2.00\text{ m}) + 6.00\frac{\text{kg}}{\text{m}}$
	$-4.00\frac{\text{kg}}{\text{m}} = slope \cdot (2.00\text{ m})$
	$slope = -2.00\frac{\text{kg}}{\text{m}^2}$

$$\mu(y) = -2.00\frac{\text{kg}}{\text{m}^2} \cdot y + 6.00\frac{\text{kg}}{\text{m}}$$

Notice: when we have a mixed formula involving both algebraic variables and numerical parameters the algebraic parameters DON'T include units while the numerical parameters DO include units.
Note: since this is an exam question, I am ignoring sig fig rules and round #'s in my final answer to three sig figs.

3b) When wire mass is negligible compared to the hanging mass, tension is essentially constant.
Here we expect $Tension = Mg = (5000\text{ kg})\left(9.8\frac{\text{m}}{\text{s}^2}\right) = 49000\text{ N}$.
Note, since this is an exam question, I am ignoring sig fig rules and will round my final answer to three sig figs.

$$v = \frac{dy}{dt}$$

$$dt = \frac{dy}{v}$$

$$\Delta t = \int_i^f \frac{dy}{v}$$

$$\Delta t = \int_i^f \frac{dy}{\sqrt{\frac{Tension}{\mu}}}$$

$$\Delta t = \int_i^f \sqrt{\frac{\mu}{Tension}}\, dy$$

$$\Delta t = \int_i^f \sqrt{\frac{slope \cdot y + intercept}{Mg}}\, dy$$

$$\Delta t = \frac{1}{\sqrt{Mg}} \int_i^f (slope \cdot y + intercept)^{1/2}\, dy$$

Instead of doing a U-sub, let's do a MU-sub…PARTY ON!
I will do this work on the next page…

Solution to 3b continues…

$$\text{If} \quad \mu = slope \cdot y + intercept \quad \text{then} \quad d\mu = slope\, dy \quad \rightarrow \quad dy = \frac{1}{slope} d\mu$$

$$\Delta t = \frac{1}{\sqrt{Mg}} \int_i^f (slope \cdot y + intercept)^{1/2}\, dy \quad \rightarrow \quad \Delta t = \frac{1}{\sqrt{Mg}} \int_i^f \mu^{1/2} \left(\frac{1}{slope} d\mu\right)$$

$$\Delta t = \frac{1}{slope\sqrt{Mg}} \int_i^f \mu^{1/2}\, d\mu$$

$$\Delta t = \frac{1}{slope\sqrt{Mg}} \left[\frac{\mu^{3/2}}{3/2}\right]_i^f$$

$$\Delta t = \frac{2}{3 \cdot slope\sqrt{Mg}} \left[\mu^{3/2}\right]_i^f$$

$$\Delta t = \frac{2}{3 \cdot slope\sqrt{Mg}} \left(\mu_{top}^{3/2} - \mu_{bottom}^{3/2}\right)$$

$$\Delta t = \frac{2}{3\left(-2.00 \frac{\text{kg}}{\text{m}^2}\right)\sqrt{49000\ \text{kg} \cdot \frac{\text{m}}{\text{s}^2}}} \left[\left(2.00 \frac{\text{kg}}{\text{m}}\right)^{3/2} - \left(6.00 \frac{\text{kg}}{\text{m}}\right)^{3/2}\right]$$

I'm getting annoyed tracking these units. I'm going to do a quick unit check so I can stop dealing with them.

$$\frac{1}{\frac{\text{kg}}{\text{m}^2} \cdot \text{kg}^{1/2} \cdot \frac{\text{m}^{1/2}}{\text{s}}} \cdot \frac{\text{kg}^{3/2}}{\text{m}^{3/2}} = \frac{\text{m}^2 \cdot \text{s}}{\text{kg} \cdot \text{kg}^{1/2} \cdot \text{m}^{1/2}} \cdot \frac{\text{kg}^{3/2}}{\text{m}^{3/2}} = \text{s}$$

$$\Delta t = \frac{2}{3(-2.00)\sqrt{49000}} \left[(2.00)^{3/2} - (6.00)^{3/2}\right]$$

$$\boldsymbol{\Delta t = 0.01787\ \text{s} = 17.87\ \text{ms}}$$

WATCH OUT! This is the time to travel to from the bottom to the top!
Sometimes I'll be sneaky and ask for *round trip* travel time (from the bottom to the top and back).

$$\Delta t_{round\ trip} = 2\Delta t_{one\ way}$$

4) Successive resonances are heard at frequencies 65 Hz & 75 Hz.

Solution Style 1:

I notice boundary conditions are MIXED:

- displacement *antinode* at the top where the speaker is located
- displacement node at the bottom where the water restricts motion just like a wall would

For MIXED boundary conditions we know

$$h = n\frac{\lambda}{4} \text{ for } n = 1, 3, 5, \ldots \quad \rightarrow \quad \lambda = \frac{4h}{n} \text{ for } n = 1, 3, 5, \ldots$$

Plug this into $v = f\lambda$ and show

$$f_n = \frac{nv}{4h} \quad \text{for } n = 1, 3, 5, \ldots$$

Notice the interval between adjacent harmonics for MIXED boundary conditions is

$$\Delta f = f_{n+2} - f_n = \frac{(n+2)v}{4h} - \frac{nv}{4h} = \frac{v}{2h}$$

Solving this for h gives

$$\boldsymbol{h = \frac{v}{2\Delta f} = 17.00 \text{ m}}$$

Solution Style 2:

The interval between successive resonances is $\Delta f = 10$ Hz.

I will find the lowest frequency by repeatedly subtracting Δf until I can no longer get a positive frequency.

$$75 \rightarrow 65 \rightarrow 55 \rightarrow 45 \rightarrow 35 \rightarrow 25 \rightarrow 15 \rightarrow 5 \text{ Hz}$$

The fundamental ($n = 1$) harmonic is thus 5 Hz.

Notice the fundamental frequency of a MIXED boundary state is HALF of the frequency interval.

$$f_{mixed\ 1} = \frac{1}{2}\Delta f$$

For MIXED boundary conditions we know

$$h = n\frac{\lambda}{4} \text{ for } n = 1, 3, 5, \ldots \quad \rightarrow \quad \lambda = \frac{4h}{n} \text{ for } n = 1, 3, 5, \ldots$$

Plug this into $v = f\lambda$ and show

$$f_n = \frac{nv}{4h} \quad \text{for } n = 1, 3, 5, \ldots$$

Using the fundamental

$$f_1 = 5 \text{ Hz} = \frac{(1)v}{4h}$$

Solving for h once again gives $h = 17.00$ m

5a) As it oscillates, the speaker will move towards and away from the person, causing Doppler shifts.
The maximum velocity of the speaker during the osscillations is given by

$$v_{max} = \omega_0 x_{max} = \sqrt{\frac{k}{m}}\, x_{max} = \sqrt{\frac{27.0}{0.0300}}\,(0.500\ \text{m}) = 15.00\,\frac{\text{m}}{\text{s}}$$

Now use the Doppler shift equation.

$$f' = f_0 \frac{c \pm v_o}{c \mp v_s}$$

In our scenario, the observer is at rest so we can set $v_o = 0$.
The max frequency observed occurs while the speaker (source) moves *towards* the observer with speed v_{max}.

$$f'_{max} = 440\ \text{Hz}\,\frac{340\,\frac{\text{m}}{\text{s}}}{340\,\frac{\text{m}}{\text{s}} - 15\,\frac{\text{m}}{\text{s}}} = 46\underline{0}\ \text{Hz}$$

The min frequency observed occurs while the speaker (source) moves *away from* the observer with speed v_{max}.

$$f'_{min} = 440\ \text{Hz}\,\frac{340\,\frac{\text{m}}{\text{s}}}{340\,\frac{\text{m}}{\text{s}} + 315\,\frac{\text{m}}{\text{s}}} = 42\underline{1}\ \text{Hz}$$

I put part 5b on the next page…

5b) Notice we are given sound intensity levels (β) not intensity (I).
Note: unless otherwise specified, we typically assume sound waves emanate *spherically*.

$$\beta_1 = 10\ \text{dB}\log_{10}\frac{I}{I_0}$$

$$\beta_1 = 10\ \text{dB}\log_{10}\frac{\frac{\mathcal{P}_{source}}{4\pi r_1^2}}{I_0}$$

$$\frac{\beta_1}{10\ \text{dB}} = \log_{10}\frac{\frac{\mathcal{P}_{source}}{4\pi r_1^2}}{I_0}$$

I am going to leave off that unit of dB; I know it ultimately cancels and it makes writing the equations easier.

$$10^{\beta_1/10} = \frac{\frac{\mathcal{P}_{source}}{4\pi r_1^2}}{I_0}$$

At this point most of you would probably determine $\mathcal{P}_{source}$ then do a second calculation of β at the new distance.
Rather than grinding away, here is a great spot to take a ratio!

$$\frac{10^{\beta_1/10}}{10^{\beta_2/10}} = \frac{\frac{\frac{\mathcal{P}_{source}}{4\pi r_1^2}}{I_0}}{\frac{\frac{\mathcal{P}_{source}}{4\pi r_2^2}}{I_0}}$$

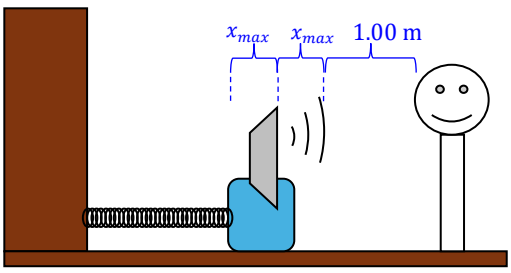
See what I mean…

$$10^{(\beta_1-\beta_2)/10} = \frac{r_2^2}{r_1^2}$$

$$\frac{\beta_1-\beta_2}{10} = \log_{10}\frac{r_2^2}{r_1^2}$$

$$\beta_2 = \beta_1 - 10\ \text{dB}\log_{10}\frac{r_2^2}{r_1^2}$$

The new distance is an *additional* $2x_{max} = 1.00$ m from the source.
Notice $r_1 = 1.00$ m & $r_2 = 2.00$ m.



$$\beta_2 = 60.0\ \text{dB} - 10\ \text{dB}\log_{10}\frac{(2.00\ \text{m})^2}{(1.00\ \text{m})^2}$$

$$\beta_2 = 53.\underline{9}8\ \text{dB}$$

Side note worth knowing for many engineers:
If you *double* intensity, *add* about 3 dB.
This stems from the fact

$$\beta_{doubled} = 10\ \text{dB}\log_{10}\left(\frac{2I}{I_0}\right) = 10\ \text{dB}\log_{10}\left(\frac{I}{I_0}\right) + 10\ \text{dB}\log_{10}2 = \beta_{original} + 3.010\ \text{dB}$$

In this case, we *divided* power by a factor of 4.
That *divides* intensity by a factor of 4.
That is cutting it in half twice → *subtract* 6 dB!

First Extra Credit (soldiers on bridge):

The *resonance* frequency is $\omega_0 = \sqrt{\frac{k}{m}} = 1.4142\,\frac{\text{rad}}{\text{s}}$.

This seems reasonable because large objects have very low fundamental frequencies.

The *driving* frequency is that of the synchronized footsteps $\left(\omega_{driver} = \omega_{dr} = 2\pi(2.00\text{ Hz}) = 12.567\,\frac{\text{rad}}{\text{s}}\right)$.

The amplitude of oscillations of a damped, driven oscillator is given by

$$A = \frac{F_0}{\sqrt{m^2(\omega_{dr}^2 - \omega_0^2)^2 + b^2\omega_{dr}^2}}$$

While not *exactly* true, we expect maximum amplitude occurs when $\omega_{dr} \approx \omega_0$.

We do not require extreme precision here so let's assume this result is exact.

Notice max amplitude is thus

$$A_{max} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega_0^2)^2 + b^2\omega_0^2}}$$

$$A_{max} = \frac{F_0}{b\omega_0}$$

Now use a ratio:

$$\frac{A_{1.41}}{A_{max}} = \frac{\frac{F_0}{\sqrt{m^2(\omega_{dr}^2 - \omega_0^2)^2 + b^2\omega_{dr}^2}}}{\frac{F_0}{b\omega_0}}$$

$$\frac{A_{1.41}}{A_{max}} = \frac{b\omega_0}{\sqrt{m^2(\omega_{dr}^2 - \omega_0^2)^2 + b^2\omega_{dr}^2}}$$

Plug in numbers from there. Note: convert this ratio to a percent to meet the requirements of the question.

Second Extra Credit (damped oscillator):

For a damped oscillator we know

$$x(t) = A\exp\left(-\frac{bt}{2m}\right)\cos(\omega_d t + \phi) \quad \text{where} \quad \omega_d = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

Notice the oscillation period is *not* $\mathbb{T} = \frac{2\pi}{\omega_0}$ but rather $\mathbb{T}_{damped} = \frac{2\pi}{\omega_d}$!

Let us assume $\phi = 0$ to simplify the computation.

This corresponds to stretching the spring to the right and releasing from rest (seems reasonable).

The initial energy is thus all spring energy given by $E_i = \frac{1}{2}kA^2$.

After on oscillation, the spring is again all the way to the right, this time with a slightly smaller amplitude.

The final energy is thus $E_f = \frac{1}{2}kA'^2$.

Again, let's take a ratio

$$\frac{A'}{A} = \frac{A\exp\left(-\frac{b\mathbb{T}_{damped}}{2m}\right)\cos\left(\omega_d\mathbb{T}_{damped}\right)}{A\exp\left[-\frac{b(0)}{2m}\right]\cos[\omega_d(0)]}$$

$$\frac{A'}{A} = \exp\left(-\frac{b\mathbb{T}_{damped}}{2m}\right)\cos\left(\omega_d\mathbb{T}_{damped}\right)$$

$$\frac{A'}{A} = \exp\left(-\frac{b\pi}{m\omega_d}\right)\cos(2\pi)$$

$$\frac{A'}{A} = \exp\left(-\frac{b\pi}{m\omega_d}\right)$$

Our requested ratio was the energy ratio

$$\frac{E_f}{E_i} = \frac{\frac{1}{2}kA'^2}{\frac{1}{2}kA^2}$$

$$\frac{E_f}{E_i} = \left(\frac{A'}{A}\right)^2$$

$$\frac{E_f}{E_i} = \exp\left(-\frac{2b\pi}{m\omega_d}\right)$$

$$\frac{E_f}{E_i} = \exp\left(-\frac{2b\pi}{m\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}\right)$$