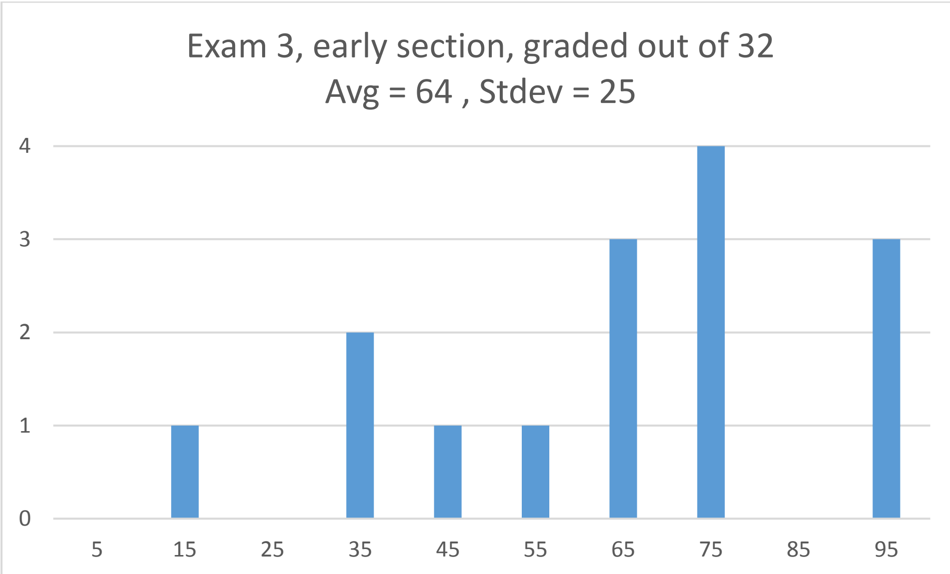
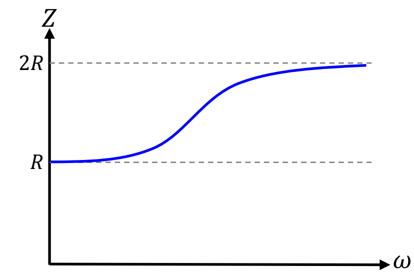
163fa21t3aSoln

Distribution on this page. Solutions begin on the next page.



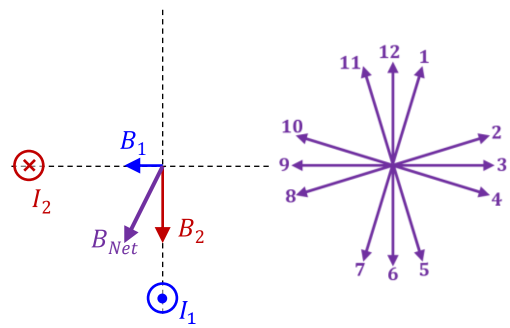
**1) At *low* frequencies the inductor acts like a short.
The two resistors with resistance $2R$ are effectively in parallel with total impedance $Z \approx R$.



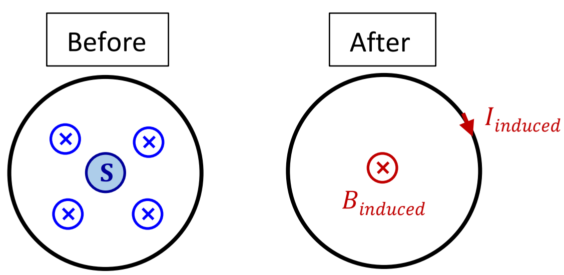
At high frequencies the inductor acts like a break.
No current flows through the right branch causing total impedance $Z \approx 2R$.

My guess is shown at right. My guess may not have exactly the correct shape, but we do know it has the correct asymptotic limits. If you care, you could do work similar to the stuff shown on page 211 in the workbook (or create a simulation) to get a more realistic shape for the curve…

*2) **Arrow seven** best represents the field direction.
WATCH OUT!
Because $i_2 = 2i_1$ we also know $B_2 = 2B_1$…
I got the directions of B_1 & B_2 from the "grab that wire" right hand rule.



3) Initially, the south pole of the magnet creates a large flux into the page through the plane of the loop.
After it moves away (farther into the page), the flux into the page decreases.
The induced EMF opposes the change in flux…
In this case, the induced EMF will attempt to replace the lost flux into the page.
Current must flow clockwise.



4a) Loop perimeter, number of turns, and total wire length are related by

$$L = N \times perimeter$$

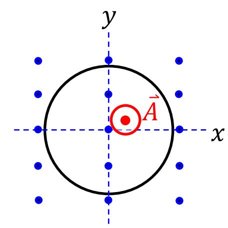
For a circular loop

$$L = N(2\pi r)$$

$$r = \frac{L}{2\pi N}$$

$$\boldsymbol{r = 10.61\ \mathrm{mm}}$$

4b) Notice, in the initial state shown in the figure at right, the loop starts from rest ($\omega_i = 0$) with the area vector aligned with the external magnetic field ($\theta_i = 0$).



For a uniform magnetic field the induced EMF equation becomes

$$EMF = -N\frac{d}{dt}B_{ext}A\cos\theta$$

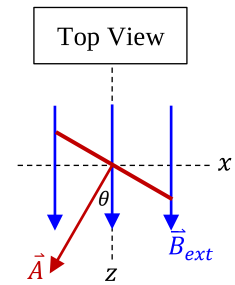
where θ is the angle between $\vec{B}_{ext}$ & $\vec{A}$.

From the constant angular acceleration kinematics we can write

$$\theta = \omega_i t + \frac{1}{2}\alpha t^2$$

Since initially at rest

$$\theta = \frac{1}{2}\alpha t^2$$



From there one finds

$$EMF = -NB_{ext}A\frac{d}{dt}\cos\theta$$

$$EMF = -NB_{ext}A(-\sin\theta)\left(\frac{d\theta}{dt}\right)$$

$$EMF = NB_{ext}A\alpha t\sin\left(\frac{1}{2}\alpha t^2\right)$$

$$EMF = NB_{ext}\pi r^2\alpha t\sin\left(\frac{1}{2}\alpha t^2\right)$$

$$EMF = NB_{ext}\pi\left(\frac{L}{2\pi N}\right)^2\alpha t\sin\left(\frac{1}{2}\alpha t^2\right)$$

$$\boldsymbol{EMF = \frac{B_{ext}L^2\alpha t}{4\pi N}\sin\left(\frac{1}{2}\alpha t^2\right)}$$

At $t = 4.44$ s one finds $\omega = \alpha t = 98.57$ rad and $\theta = \frac{1}{2}\alpha t^2 = 218.8$ rad.

$$\boldsymbol{EMF = -78\underline{0}\ \mathrm{mV}}$$

Verify you used radians mode in the calculator when doing the trig function (or converted to $\boldsymbol{\theta = 12536°}$)!

Alternate style:

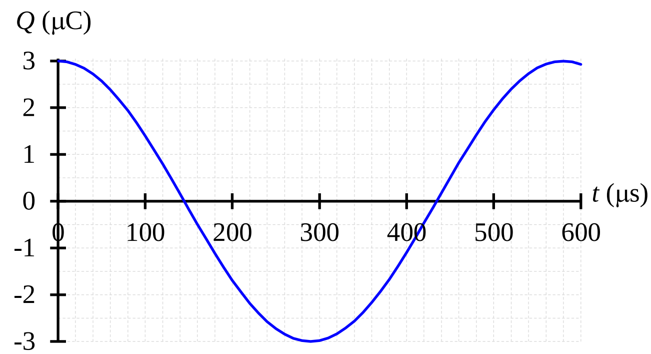
$$EMF = -\omega NB_{ext}A\sin\theta$$

$$EMF = -(\alpha t)NB_{ext}(\pi r^2)\sin\left(\frac{1}{2}\alpha t^2\right)$$

5a) From the plot at right one can determine:

- period is $\mathbb{T} = 580\ \mu\text{s}$
- max charge is $Q_{max} = 3.00\ \mu\text{C}$

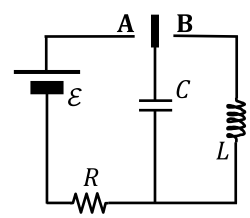
We probably can't get 3 sig figs off this graph, but for exam purposes we typically assume three sig figs on all numbers.



The problem statement tells us $C = 250.0\ \text{nF}$.
Note: we immediately know

$$\omega_0 = \frac{2\pi}{\mathbb{T}} = 10.83\frac{\text{krad}}{\text{s}}$$

Watch out!
Most students poorly estimated the period by assuming the first zero of the trig function occurs at $140\ \mu\text{s}$.
This was acceptable on test day but is a bad practice in general.
Next time, use the entire cycle to estimate the period to reduce the % error associated with reading the plot…



When left in position **A** for a long time, we know the capacitor reaches max charge given by

$$Q_{max} = \mathcal{E}C$$

Therefore

$$\mathcal{E} = \frac{Q_{max}}{C} = \mathbf{12.00\ V}$$

5b) The inductor first has max energy when the capacitor first has zero energy (zero charge).
This occurs at time $145\ \mu\text{s}$. I probably should take off points for $140\ \mu\text{s}$ or $150\ \mu\text{s}$ because, in my opinion, the period is *clearly* $580\ \mu\text{s}$… I didn't since this was a test situation; reading plots naturally introduces small errors.

5c) Resonance (angular) frequency is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Rearranging and solving for L gives

$$L = \frac{1}{\omega_0^2 C} = \mathbf{34.1\ mH}$$

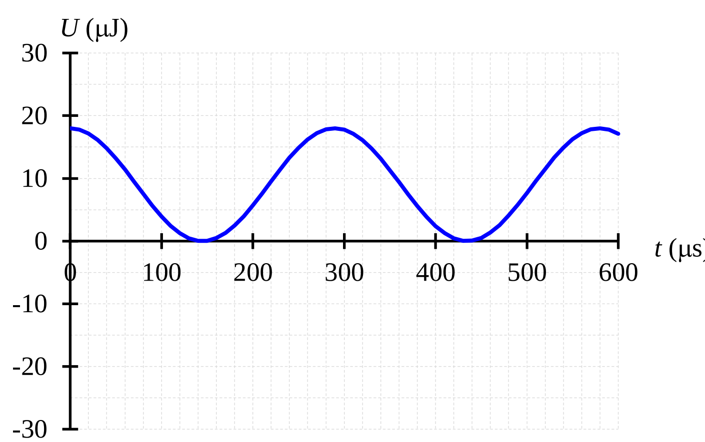
5d) The energy never goes negative and oscillates twice as fast. You can see this by first writing down

$$U(t) = \frac{[Q(t)]^2}{2C}$$

$$U(t) = \frac{Q_{max}^2}{2C}\cos^2(\omega_0 t)$$

$$U(t) = \frac{Q_{max}^2}{2C}\left(\frac{1+\cos 2\omega_0 t}{2}\right)$$

Side note: for LC oscillator circuits, oscillations occur at resonance frequency (no function generator present).
For LRC series circuit, one uses function generator *operating* frequency ω inside the trig functions…
Note: one can also produce this plot by squaring the numbers on the given plot, dividing by $2C$, then sketching & connecting the dots…



6a) $[k] = \frac{\mathrm{A}}{\mathrm{m}^5}$
6b) One finds

$$I = \int_{R/2}^{R} J\, dA$$

$$I = \int_{R/2}^{R} (k\tilde{r}^3)\; 2\pi\tilde{r}\; d\tilde{r}$$

$$I = 2\pi k \left[\frac{\tilde{r}^5}{5}\right]_{R/2}^{R}$$

From there I found

$$\boldsymbol{k = \frac{80I}{31\pi R^5} \approx 0.821\frac{I}{R^5}}$$

6c) into the page
6d) 0
6e) Use

$$Bs = \mu_0 I_{enclosed}$$

$$B(2\pi r) = \mu_0 \int_{R/2}^{r} (k\tilde{r}^3)\; 2\pi\tilde{r}\; d\tilde{r}$$

$$B(2\pi r) = \mu_0 2\pi k \left[\frac{\tilde{r}^5}{5}\right]_{R/2}^{r}$$

$$\boldsymbol{B = \frac{\mu_0 k}{5r}\left(r^5 - \frac{R^5}{32}\right)}$$

While you could stop there, it is often easier to understand and compare to other fields if we replace k using

$$B = \frac{\mu_0\left(\frac{\mathbf{80\, I}}{\mathbf{31\pi\, R^5}}\right)}{5r}\left(r^5 - \frac{R^5}{32}\right)$$

$$\boldsymbol{B = \frac{16\,\mu_0 I}{31\pi\, r R^5}\left(r^5 - \frac{R^5}{32}\right)}$$

One could also clean up the numerical factors and bring R^5 inside the paren's:

$$\boldsymbol{B = 0.1643\frac{\mu_0 I}{r}\left(\left(\frac{r}{R}\right)^5 - \frac{1}{32}\right)}$$

6f) Outside is simple. We know total current from the problem statement.

$$\boldsymbol{B = \frac{\mu_0 I}{2\pi r}}$$

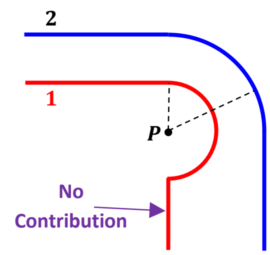
To check these results, we can compare the solutions for regions 2 & 3 by plugging in $\boldsymbol{r = R}$.

$$B = \frac{16\mu_0 I}{31\pi (R) R^5}\left(R^5 - \frac{R^5}{32}\right)$$

$$B = \frac{16\,\mu_0 I}{31\pi\, R^6}\cdot\frac{31}{32}R^5$$

$$B = \frac{\mu_0 I}{2\pi R}$$

****7) The scenario is only possible if currents run opposite directions (so magnetic fields created by each wire point in opposite directions at **P**).
If this is true, the magnetic field magnitude created by each wire must to be the same at **P**.



$$B_1 = mag\ field\ MAGNITUDE = \frac{1}{2}(\infty\ wire) + \frac{1}{2}(circular\ loop\ at\ center)$$

$$B_1 = \frac{1}{2}\left(\frac{\mu_0 I}{2\pi r_1}\right) + \frac{1}{2}\left(\frac{\mu_0 I}{2r_1}\right)$$

$$B_1 = \frac{\mu_0 I}{r_1}\left(\frac{1}{4\pi} + \frac{1}{4}\right)$$

$$B_1 = 0.3296\frac{\mu_0 I}{r_1}$$

$$B_2 = mag\ field\ MAGNITUDE = \frac{1}{2}(\infty\ wire) + \frac{1}{2}(\infty\ wire) + \frac{1}{4}(circular\ loop\ at\ center)$$

$$B_2 = mag\ field\ MAGNITUDE = (\infty\ wire) + \frac{1}{4}(circular\ loop\ at\ center)$$

$$B_2 = \left(\frac{\mu_0(2I)}{2\pi r_2}\right) + \frac{1}{4}\left(\frac{\mu_0(2I)}{2r_2}\right)$$

WATCH OUT! This set of segments uses *twice the current* ($I \rightarrow 2I$)…

$$B_2 = \frac{\mu_{02}I}{r_2}\left(\frac{1}{\pi} + \frac{1}{4}\right)$$

$$B_2 = 0.5683\frac{\mu_0 I}{r_2}$$

Now set the magnitudes equal and solve for the ratio as requested

$$B_1 = B_2$$

$$0.3296\frac{\mu_0 I}{r_1} = 0.5683\frac{\mu_0 I}{r_2}$$

$$\frac{r_2}{r_1} = \frac{0.5683}{0.3296}$$

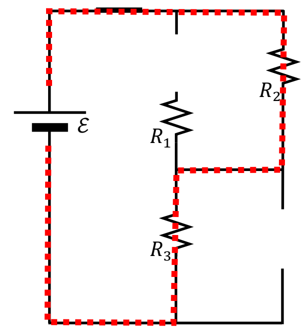
$$\frac{r_2}{r_1} = 1.724$$

Notice I kept extra sig figs on the intermediate results to avoid intermediate rounding error.

8a) Just after the switch is closed, the inductor acts as a break.
Current flows through resistors R_2 & R_3 in series.
The figure at right shows the $t = 0_+$ current path with a dotted red line.
Notice current through R_3 is the same as current through the battery with equivalent resistance

$$R_{eq} = R_2 + R_3 = 2R$$

$$I_3 = I_{battery} = \frac{\mathcal{E}}{R_{eq}} = \frac{\mathcal{E}}{2R}$$



8b) When the circuit reaches steady state, the two inductors acts like shorts.
In steady state, we see resistors R_1 & R_2 are in parallel while resistor R_3 is bypassed by the inductor (which is acting like a short).
The figure at right shows the steady state current path with a red dotted line.

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} = \frac{R}{2}$$

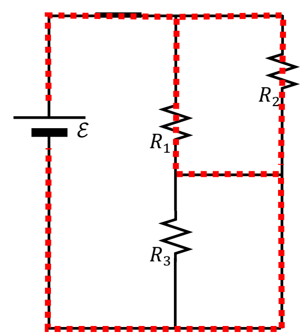
Power is given by

$$\mathcal{P} = I\Delta V$$

Since we know $\Delta V_{battery} = \mathcal{E}$ & $R_{eq} = \frac{R}{2}$ I choose to rewrite power equation in terms of voltage & resistance.

$$\mathcal{P}_{battery} = \frac{\left(\Delta V_{battery}\right)^2}{R_{eq}}$$

$$\boldsymbol{\mathcal{P}_{battery} = \frac{2\mathcal{E}^2}{R}}$$



9a) We are told the following info:

- $V_0 = source\ voltage\ amplitude = 7.77$ V
- $f_{operating} = 1.88$ kHz (NOTE: $\omega_{operating} = \omega = 2\pi f_{operating} = 118\underline{1}2\,\frac{\text{rad}}{\text{s}}$)
- $R = 66.6\ \Omega$
- $C = 555$ nF
- $\mathcal{P}_{avg} = 444$ mW

Average power relates to current amplitude (i_{max}) & resistance…but you must do a little work to get there.

$$\mathcal{P}_{avg} = i_{RMS}^2 R$$

$$\mathcal{P}_{avg} = \left(\frac{i_{max}}{\sqrt{2}}\right)^2 R$$

$$\mathcal{P}_{avg} = \frac{i_{max}^2}{2} R$$

$$i_{max} = \sqrt{\frac{2\mathcal{P}_{avg}}{R}}$$

$$\boldsymbol{i_{max} = 115.5\ \mathbf{mA}}$$

9b) The formula for total impedance (Z) includes the unknown inductance as well as operating frequency, capacitance, and resistance (all known).
Furthermore, Z relates to current amplitude (i_{max}) and source voltage amplitude (V_0).

$$i_{max} = \frac{V_0}{Z}$$

$$Z = \frac{V_0}{i_{max}}$$

$$\sqrt{R^2 + (X_L - X_C)^2} = \frac{V_0}{i_{max}}$$

$$R^2 + (X_L - X_C)^2 = \left(\frac{V_0}{i_{max}}\right)^2$$

$$(X_L - X_C)^2 = \left(\frac{V_0}{i_{max}}\right)^2 - R^2$$

$$X_L - X_C = \sqrt{\left(\frac{V_0}{i_{max}}\right)^2 - R^2}$$

$$\omega L = \frac{1}{\omega C} + \sqrt{\left(\frac{V_0}{i_{max}}\right)^2 - R^2}$$

$$L = \frac{1}{\omega^2 C} + \frac{1}{\omega}\sqrt{\left(\frac{V_0}{i_{max}}\right)^2 - R^2}$$

$$\boldsymbol{L = 13.72\ \mathbf{mH}}$$

Extra Credit:

The loop has both resistance and inductance with impedance given by

$$Z = \sqrt{R^2 + (\omega L)^2}$$

and phase angle given by

$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$\phi = \tan^{-1}\left(\frac{\alpha t L}{R}\right)$$

Phase angle is initially zero but it should gradually increase with time.