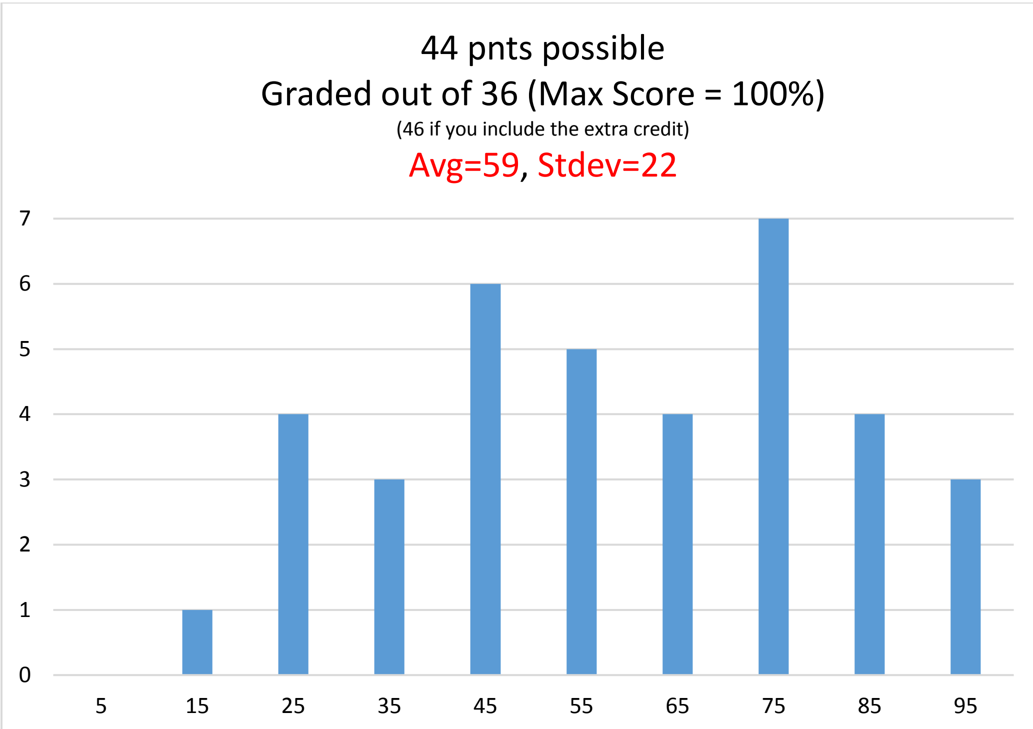
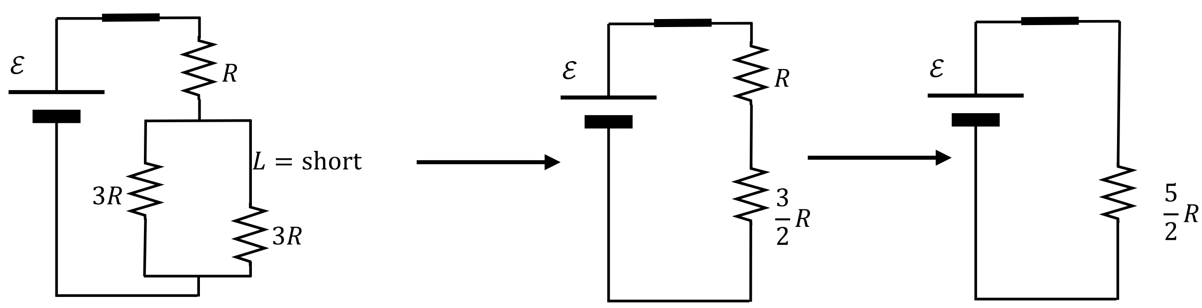
163fa22t3aSoln

Distribution on this page; solutions begin on the next page.



1a) Just after the switch is closed, the inductor preserves its previous state (zero current). Since the bottom right resistor is in series with the inductor, it also has zero current.

1b) **ANSWER is $\boldsymbol{\frac{\mathcal{E}}{5R}}$.**
After a long time with the switch in the closed position, the inductor acts like a short. The circuit can be redrawn as shown in the set of figures below.

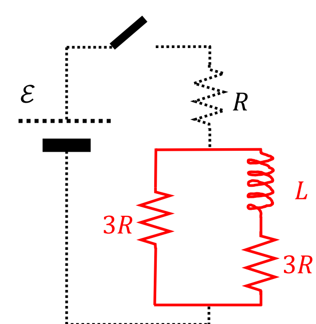


Notice total current through the battery is $\frac{2\mathcal{E}}{5R}$.
Only half of this current flows through the bottom right resistor.

1c) **ANSWER is $\boldsymbol{\frac{\mathcal{E}}{5R}}$.**
Upon re-opening the switch, the inductor preserves its current from the instant before the switch is re-opened.
In this case, that implies the current in the inductor remains at $\frac{\mathcal{E}}{5R}$.

1d) Upon reopening the switch, the circuit is effectively reduced to the loop containing the inductor and the two $3R$ resistors (shown in red at right).
Notice, the two resistors are now in *series*!
The time constant is

$$\boldsymbol{\tau = \frac{L}{6R}}$$



1e) For this scenario:

- The $t = 0_-$ state is switch closed for a long time.
- The $t = 0_+$ state is just after the switch is re-opened.
- The $t = \infty$ state is after the inductor has been fully drained of energy.

The *initial* current through each resistor in series with the inductor is

$$i = \frac{\mathcal{E}}{5R}$$

The *initial* voltage across the bottom right resistor is

$$\Delta V_{3R} = i(3R)$$

$$\Delta V_{3R} = \frac{3}{5}\mathcal{E}$$

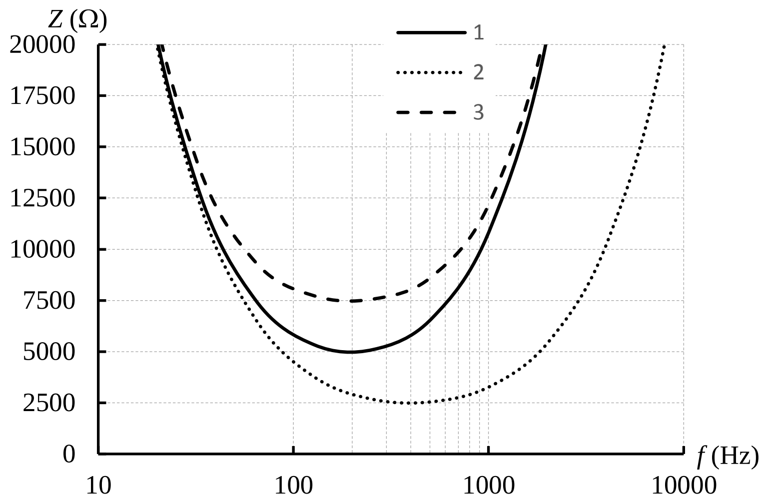
The final voltage across all circuit elements is zero.

$$\Delta V_{3R}(t) = \frac{3}{5}\mathcal{E}\exp\left(-\frac{6Rt}{L}\right)$$

Note: if you are unfamiliar with what exp() means…it is simply a different way of writing $e^{()}$.
It is used whenever the term n the exponent is so busy things are hard to read.
In this case $\exp\left(-\frac{6Rt}{L}\right) = e^{-6Rt/L}$.

2a) **Capacitors** dominate the circuit at low frequencies.
I remember it this way, at low frequencies, the capacitors have ample time to almost fully charge (thus dramatically impeding current flow).

2b) We know impedance is a minimum at resonance (at resonance $X_L = X_C$ giving $Z = R$). **Circuits 1 & 3 have the lowest resonance frequencies.**



2c) We know the resonance (angular) frequency condition is

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Rearranging gives

$$L = \frac{1}{\omega_0^2 C}$$

We were told all circuits employ identical C.
Largest resonance frequency implies smallest L.

$$\boldsymbol{L_2 < L_1 = L_3}$$

2d) We know impedance is a minimum at resonance (at resonance $X_L = X_C$ giving $Z = R$).

R_1 (kΩ)	R_2 (kΩ)	R_3 (kΩ)
5.00	2.50	7.50

3a) I used

$$I = \int_R^{2R} \vec{J} \cdot d\vec{A}$$

$$I = \int_R^{2R} \frac{\alpha}{\tilde{r}^3}\, 2\pi\tilde{r}\, d\tilde{r}$$

$$I = 2\pi\alpha \left[-\frac{1}{\tilde{r}}\right]_R^{2R}$$

From there I solved for α and found

$$\boldsymbol{\alpha = \frac{IR}{\pi}}$$

3b) For the magnetic field to be zero in the outermost region (region 5) we know total current enclosed must be zero. In this case we know the current in the outer shell must be the same size as the inner shell (I) running in the opposite direction (out of the page).

Since this current is uniformly distributed we know

$$\vec{J} = \frac{I}{\pi(5R)^2 - \pi(3R)^2}\hat{k}$$

$$\vec{J} = \frac{I}{16\pi R^2}\hat{k}$$

Note: technically, current density is a vector so I suppose we should probably include the $\hat{k}$.
This might be the only instance I can think of where I'd let you omit direction (even though I probably shouldn't).

3c) No current enclosed, **no mag field**.

3d) I used

$$B_1(2\pi r) = \mu_0 \int_R^{r} \frac{\alpha}{\tilde{r}^3}\, 2\pi\tilde{r}\, d\tilde{r}$$

I wrote up the final answer in two forms

$$B_1 = \frac{\mu_0\alpha}{r}\left(\frac{1}{R} - \frac{1}{r}\right) \qquad \text{or} \qquad \boldsymbol{B_1 = \frac{\mu_0 I}{2\pi r} \cdot 2\left(1 - \frac{R}{r}\right)}$$

I like the second form (without the $\frac{\mu_0 I}{2\pi r}$ factored out) because it makes it easy to check the boundary condition against the answer to the next part. All I did was back sub in the value for α determine in part a…

3e) Don't over think it. We are outside a cylindrical object carrying total current I.

$$B_2 = \frac{\mu_0 I}{2\pi r}$$

Do be careful to distinguish between r (a variable radial position between the two shells) and R or $2R$ (the constant inner and outer radii of the inner shell).

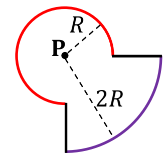
3f) I used

$$B_4(2\pi r) = B_1 - \mu_0 I \frac{A_{of\ outer\ shell\ enclosed\ by\ Amperian\ loop}}{A_{total\ of\ outer\ shell}}$$

$$B_4 = \frac{\mu_0 I}{2\pi r}\left(1 - \frac{r^2 - 9R^2}{16R^2}\right)$$

Again I choose this form since it makes it easy to compare to the previous part's answer at the boundary.

4a) Out of the page magnetic field at **P** implies **Counter-Clockwise** current.



4b) Out of the page magnetic field at **P′** implies current **to the right** in the horizontal segment.

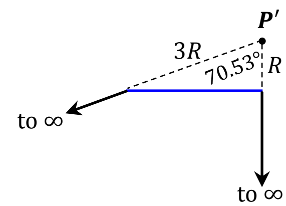
4c) Looking at the upper coil, we have ¾ of a small circle plus ¼ of a big circle.
The straight line segments produce zero contribution at **P** since they are directed radially away and radially towards.

$$B_{upper} = \frac{1}{4}\cdot\frac{\mu_0 i}{2(2R)} + \frac{3}{4}\cdot\frac{\mu_0 i}{2(R)}$$

$$B_{upper} = \frac{7}{16}\cdot\frac{\mu_0 i}{R} = 0.4375\frac{\mu_0 i}{R}$$

We were told $B_{upper} = B$ and to solve for i.

$$\boldsymbol{i = \frac{16}{7}\cdot\frac{BR}{\mu_0} \approx 2.29\frac{BR}{\mu_0}}$$



4d) In the lower wire only the horizontal segment contributes to the field at **P′**.
The segments going out to ∞ produce zero contribution at **P′** since they are directed radially away and radially towards.

$$B_{lower} = \frac{\mu_0 i}{4\pi a}\left(\sin\theta_f - \sin\theta_i\right)$$

In this expression we know a is the pistance from **P′** to the wire along the perpendicular bisector (distance R).
Angles are measured from **P′** to the ends of the wire segment with the angle to the perpendicular bisector being zero.
Using the triangle with sides R & $3R$ and SOH CAH TOA one finds the $70.53°$ angle shown in the figure.

$$B_{lower} = \frac{\mu_0 i}{4\pi R}\left(\sin 70.\underline{5}3° - \sin 0°\right) \approx 0.075\underline{0}3\frac{\mu_0 i}{R}$$

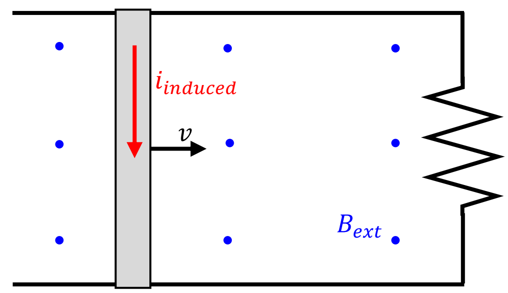
Taking the ratio gives

$$ratio = \frac{B_{upper}}{B_{lower}}$$

$$ratio = \frac{0.4375\frac{\mu_0 i}{R}}{0.075\underline{0}3\frac{\mu_0 i}{R}}$$

$$\boldsymbol{ratio \approx 5.83}$$

5a) As the rod moves it reduces flux out of the page through the loop. Induced current should try to replace the lost out of the page flux by producing a field that is also out of the page.
Current in the loop must flow counter-clockwise.
In the rod current flows downwards.



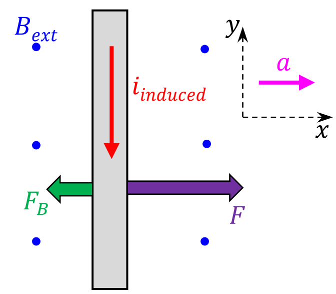
5b) We want current as a function of time.

$$i_{induced} = \frac{\mathcal{E}_{motional}}{R} = \frac{B_\perp L v}{R}$$

All we need to do is determine $v(t)$ and we have it…

Consider the FBD of the rod shown below at right.
I included $i_{induced}$ and B_{ext} on the figure so you could see how a right hand rule gives magnetic force $\vec{F}_B$ to the left.

FBD of Rod



Why is the arrow for $\vec{F}_B$ smaller than the arrow for the applied force $\vec{F}$?
Initially the rod moves slowly (it accelerates *from rest*).
The induced EMF in the rod is

$$\mathcal{E}_{motional} = B_\perp L v$$

We expect a small $\mathcal{E}_{motional}$.
This produces a small $i_{induced}$.
This causes a small magnetic force magnitude $F_B = iLB_{ext}\sin(ANGLE)$.
Here $ANGLE$ is the angle between $\vec{B}_{ext}$ and $\vec{L}$ (direction of current flow).
Putting together all the pieces into a force equation gives

$$F - F_B = ma$$

$$F - iLB_{ext}\sin(ANGLE) = ma$$

$$F - iLB\sin(90°) = ma$$

$$F - \frac{\mathcal{E}_{motional}}{R} LB = ma$$

$$\boldsymbol{F - \frac{B^2L^2}{R}v = ma} \qquad \textbf{(can get } \boldsymbol{v_T} \textbf{ by setting } \boldsymbol{a = 0}\textbf{)}$$

$$\frac{F}{m} - \frac{B^2L^2}{mR}v = \frac{dv}{dt}$$

$$-\frac{B^2L^2}{mR}\left(v - \frac{FR}{B^2L^2}\right) = \frac{dv}{dt}$$

$$-\frac{B^2L^2}{mR}dt = \frac{dv}{v - \frac{FR}{B^2L^2}}$$

$$\boldsymbol{-\frac{B^2L^2}{mR}\int_{t_i}^{t_f} dt = \int_{v_i=0}^{v_f} \frac{dv}{v - \frac{FR}{B^2L^2}}} \qquad \textbf{(can get } \boldsymbol{\tau} \textbf{ using unit analysis)}$$

Solution continues on the next page…

From here the integration is straightforward. Solve for v_f which is then identified as $v(t)$.
Alternatively, one could identify this one of the special differential equations when we can use

$$X(t) = X_f + (X_i - X_f)e^{-t/\tau}$$

If using this technique, one must notice the term $\frac{B^2L^2}{mR}$ in the last line has units of $\frac{1}{\text{seconds}}$.
Since τ has units of seconds, we see $\boldsymbol{\tau = \frac{mR}{B^2L^2}}$!
Similarly, the fifth line above can be used to show $v_{Terminal} = \boldsymbol{v_T = \frac{RF}{B^2L^2}}$

$$\boldsymbol{v(t) = v_T\left(1 - e^{-\frac{t}{\tau}}\right)}$$

Going back to the start of the problem:

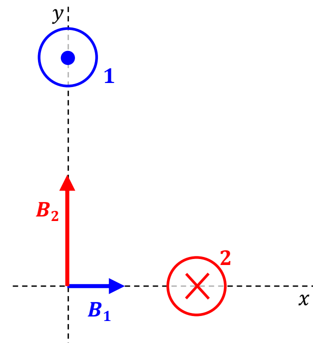
$$i_{induced} = \frac{\mathcal{E}_{motional}}{R}$$

$$i_{induced} = \frac{B_\perp Lv}{R}$$

$$i_{induced} = \frac{B_\perp L}{R} \cdot \boldsymbol{\frac{RF}{B^2L^2}\left(1 - e^{-\frac{t}{\tau}}\right)}$$

$$\boldsymbol{i_{induced}} = \boldsymbol{\frac{F}{BL}\left(1 - e^{-\frac{t}{\tau}}\right)} \quad \text{where } \boldsymbol{\tau} = \boldsymbol{\frac{mR}{B^2L^2}}$$

6abc) The wires produce magnetic fields in the directions shown in the figure.

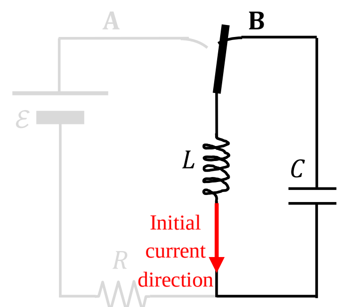


6a) $B_x > 0$

6b) $B_y > 0$

6c) $|B_x| < |B_y|$

7a) When the switch is in position **B**, we have an LC oscillator circuit.



$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\frac{2\pi}{\mathbb{T}} = \frac{1}{\sqrt{LC}}$$

$$\mathbb{T} = 2\pi\sqrt{LC}$$

$$\mathbb{T} \approx \mathbf{162.2\ \mu s}$$

7b) The inductor preserves its current as the switch is changed.
Just *before* switching, the inductor was in steady state with current flowing downwards. Just *after* switching, current still flows the same way (see figure).
This current direction implies electron flow is the opposite direction.
We expect *electrons* will initially build up on the *top* plate of the capacitor.
Initially the bottom plate of the capacitor is positively charged.

7c) With zero resistance in the LC oscillator loop, energy is conserved during oscillations (and they go on forever).
Note: in the real world, the tiny resistance of the wires does cause energy loss.
As a result, in the real world oscillations will not go on forever.
That said, if resistance in the wires is very small, the following statement is a pretty solid approximation.

$$U_{C\,max} = U_{L\,max} \quad = U$$

$$\frac{1}{2} \cdot \frac{Q_{max}^2}{C} = \frac{1}{2} \cdot L i_{max}^2 = U$$

$$Q_{max} = i_{max}\sqrt{LC} = \sqrt{2UC}$$

$$\boldsymbol{Q_{max}} \approx \mathbf{17.31\ \mu C}$$

7d) We know i_{max} in this circuit is the steady state current just before the switch was flipped $\left(i_{max} = \frac{\mathcal{E}}{R}\right)$!

$$U_{L\,max} = \frac{1}{2} \cdot L i_{max}^2$$

$$i_{max} = \sqrt{\frac{2U_{L\,max}}{L}}$$

$$\frac{\mathcal{E}}{R} = \sqrt{\frac{2U_{L\,max}}{L}}$$

$$\boldsymbol{\mathcal{E} = R\sqrt{\frac{2U_{L\,max}}{L}} \approx 315\ \mathrm{V}}$$

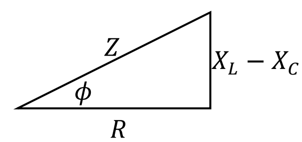
8a) **The capacitor is dominating this circuit.**
The phase angle equation is

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

A negative phase angle implies $X_C > X_L$.
Note: this implies we must be operating at a frequency *below* resonance…

8b) **Current leads source voltage.**
We know the phrase ELI the ICE organism.
Use the word ICE because it contains the letter C (and we know the capacitor is dominating the circuit).

8c) **Increasing capacitance OR increasing function generator frequency increases current amplitude.**
Think: we know current amplitude is largest at resonance.
Increasing the function generator operating frequency should get us closer to resonance (when C dominates).
How do I know this? Recall the Note: from part a…
Think: We know capacitive reactance is $X_C = \frac{1}{\omega C}$.
If we increase capacitance, capacitive reactance decreases and the capacitor is less dominant.
When we do this, we shift *resonance* frequency closer to the *operational* frequency!
Another weird way to think of it: if we use a bigger C it gets less full of charge (percentagewise) and thus impedes current flow slightly less.



8d) I suppose I see two ways to approach this (but both require essentially the same mathematics).

- Method 1: Use the phase angle equation to find ω (and thus f).
- Method 2: Use the phase triangle to solve for Z. Use this Z to determine ω (and thus f).

HEY YOU! Remember the distinction between resonance frequency $\left(\omega_0 = \frac{1}{\sqrt{LC}}\right)$ and operational frequency (ω)!
SIDE NOTE: I find it easier to first solve for ω to avoid dealing with a crapload of 2π's in my work.

$$\tan\phi = \frac{X_L - X_C}{R}$$

$$R\tan\phi = X_L - X_C$$

$$R\tan\phi = \omega L - \frac{1}{\omega C}$$

Multiply all terms by ωC to get rid of the fraction…

$$\omega RC\tan\phi = \omega^2 LC - 1$$

Divide all by LC to simplify the ensuing quadratic formula (also easier for checking units IMHO).

$$\omega\frac{R}{L}\tan\phi = \omega^2 - \frac{1}{LC}$$

$$\omega\frac{R}{L}\tan\phi = \omega^2 - \omega_0^2$$

$$0 = \omega^2 - \omega\frac{R}{L}\tan\phi - \omega_0^2$$

Use quadratic formula with $a = 1$, $b = -\frac{R}{L}\tan\phi$, and $c = -\omega_0^2$.

$$\omega = \frac{-(-\frac{R}{L}\tan\phi) \pm \sqrt{\left(-\frac{R}{L}\tan\phi\right)^2 - 4(-\omega_0^2)(1)}}{2(1)}$$

Continues on the next page…

Sometimes I like to reduce the clutter by moving that 2 into every term in the numerator.
Notice this requires dividing by 4 for terms inside the square root…

$$\omega = \frac{R}{2L}\tan\phi \pm \sqrt{\frac{R^2}{4L^2}\tan^2\phi + \omega_0^2} = \frac{R}{2L}\tan\phi\left(1 \pm \sqrt{+\frac{\omega_0^2}{\frac{R^2}{4L^2}\tan^2\phi}}\right)$$

Side math to simplify computation:

$\frac{R}{2L}\tan\phi = -1.192511 \times 10^5 \frac{\text{rad}}{\text{s}}$ I recognize the units as being $\frac{1}{\text{s}}$ because I know the time constant from RL transients is $\tau = \frac{L}{R}$…	$\frac{R^2}{4L^2}\tan^2\phi = 1.422083 \times 10^{10} \frac{\text{rad}^2}{\text{s}^2}$	$\omega_0^2 = \frac{1}{LC} = 1.5015015 \times 10^9 \frac{\text{rad}^2}{\text{s}^2}$

WATCH OUT FOR THAT MINUS SIGN! The phase angle was negative…

Notice I get the correct units of $\frac{\text{rad}}{\text{s}}$ on the output ω.

Because I notice this, I feel confident leaving units off my intermediate work…

$$\omega = -1.192511 \times 10^5 \pm \sqrt{1.422083 \times 10^{10} + 1.5015015 \times 10^9}$$

$$\omega = -1.192511 \times 10^5 \pm 1.253887 \times 10^5$$

YIKES! Notice the subtraction above causes a loss of two sig figs!

Thus the hint in the problem statement to include extra sig figs on your intermediate answers.

In this case, use the positive root since we require $\omega > 0$.

$$\omega = 6.138 \times 10^3 \frac{\text{rad}}{\text{s}}$$

Check: with a negative phase angle, the result for ω had better be less than $\omega_0 = \frac{1}{\sqrt{LC}} = 3.87 \times 10^4 \frac{\text{rad}}{\text{s}}$…it is!

Now I can divide this result by 2π to get the frequency in units of Hz.

$$\boldsymbol{f = 977\ \text{Hz}}$$