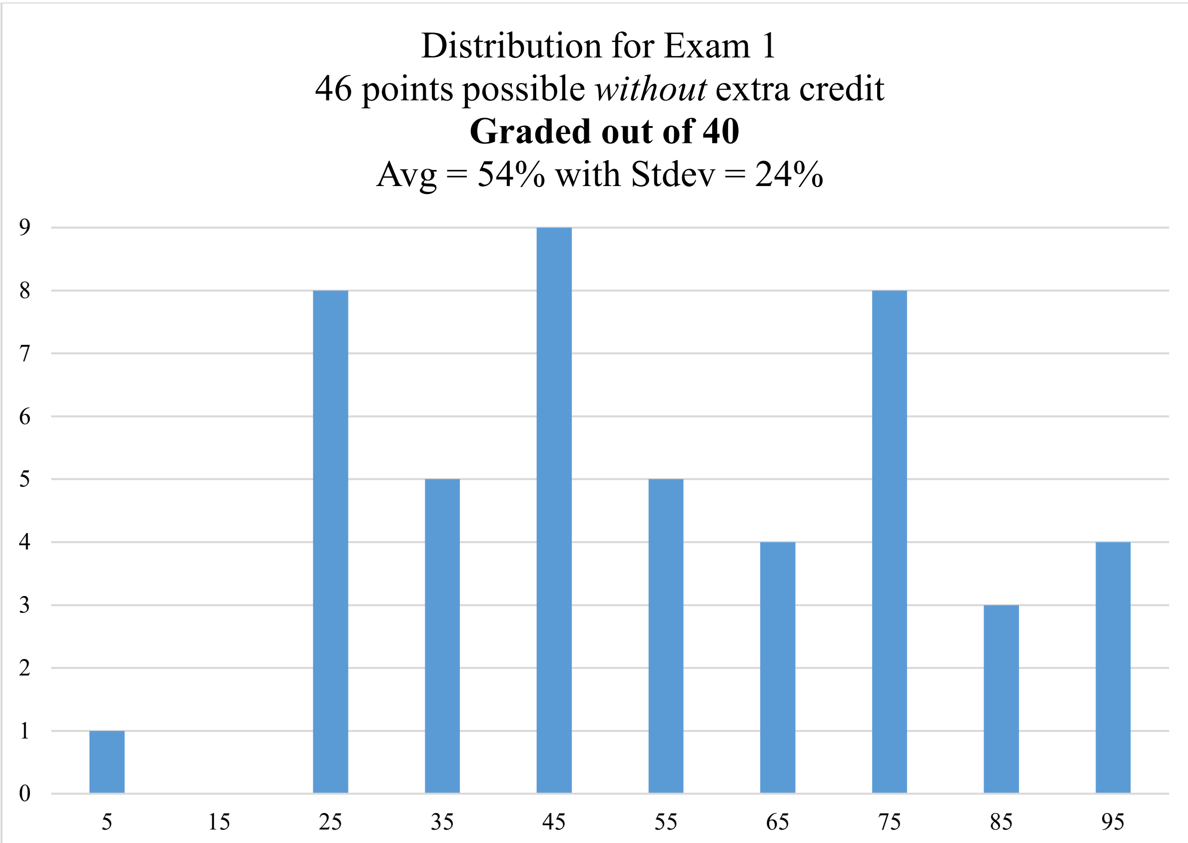
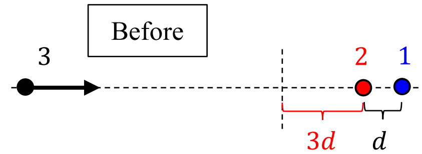
163fa23t1aSoln

Distribution on this page.
Solutions begin on the next page.



1a) Do a conservation of energy problem.

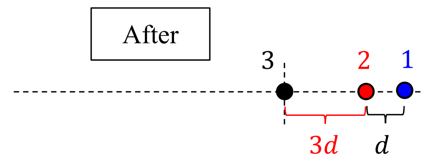
$$K_i + U_i = K_f + U_f$$



$$\frac{1}{2}mv^2 + \frac{kq_1q_2}{d} = 0 + \left(\frac{kq_1q_2}{d} + \frac{kq_1q_3}{4d} + \frac{kq_2q_3}{3d}\right)$$

To save time plugging in I notice the cancellation of $\frac{kq_1q_2}{d}$…

$$\frac{1}{2}mv^2 = \frac{kq_1q_3}{4d} + \frac{kq_2q_3}{3d}$$

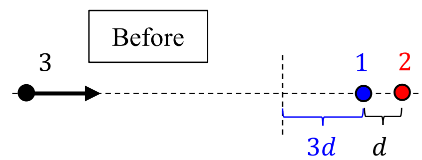


$$\frac{1}{2}mv^2 = \frac{k(-e)(e)}{4d} + \frac{k(2e)(e)}{3d}$$

$$\frac{1}{2}mv^2 = \frac{ke^2}{d}\left(-\frac{1}{4} + \frac{2}{3}\right)$$

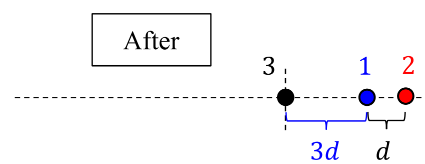
$$\boldsymbol{v = \sqrt{\frac{5ke^2}{6md}} \approx 0.913\sqrt{\frac{ke^2}{md}}}$$

1b) What if the positions of the two fixed charges are reversed?
Previously fixed charge with *same* sign as q_3 was slightly closer to q_3.
In the new situation, the fixed charge with *opposite* sign to q_3 is slightly closer.
It should make it slightly easier to reach the origin.
Less initial speed should be required to reach the origin.



If you are curious, the math changes to

$$\frac{1}{2}mv^2 = \frac{k(-e)(e)}{3d} + \frac{k(2e)(e)}{4d}$$



$$\frac{1}{2}mv^2 = \frac{ke^2}{d}\left(-\frac{1}{3} + \frac{1}{2}\right)$$

$$\boldsymbol{v = \sqrt{\frac{ke^2}{3md}} \approx 0.577\sqrt{\frac{ke^2}{md}}}$$

2a) The electric field is given in the code in line 4.

LINE 15	`F_elec = ball.q * E_field`

2b) Generally speaking, it is *preferential* (but not required) to compute constant quantities OUTSIDE of the loop.

If you are a CS major and you defined constant quantities inside the loop I suppose I *should* say:

Shame on you.

That said, if you put everything in the loop and wrote a functional code I am actually pretty happy.

Line #	Code
17	`F_net = F_elec + F_grav`
18	`ball.a = F_net / ball.m`
26	`ball.v += ball.a * dt`
27	`ball.pos += ball.v * dt`
28	`t += dt`

Think: if you don't remember to increment time the loop while never end!!!

```
1  Web VPython 3.2
2
3  #E_field is a UNIFORM electric field vector
4  E_field = vec(7.832, -4.516, -6.097)
5  ball = sphere()
6  ball.m = 1.2345     #mass in units of kg
7  ball.q = -3.0548    #charge in units of C
8  ball.v = vec(0,0,0) #velocity in m/s
9  ball.a = vec(0,0,0) #acceleration in m/s**2
10
11 k = 8.99e9          #in units of N*m**2/C**2
12 g = 9.8             #in unis of m/s**2
13
14 #compute forces
15 F_elec =
16 F_grav = vec(0, -ball.m*g, 0)
17
18
19
20
21 t=0
22 dt=0.1
23 sim_speed = 1
24 while t<5:
25     rate(sim_speed/dt)
26
27
28
29
```

3a) The steepest section of the curve is in bold (50 → 55 nm).

$$F_x = qE_x$$

$$F_x = (-e)\left(-\frac{dV}{dx}\right)$$

$$F_x = (-e)(-slope)$$

Minus signs cancel!

$$F_x = e(slope)$$

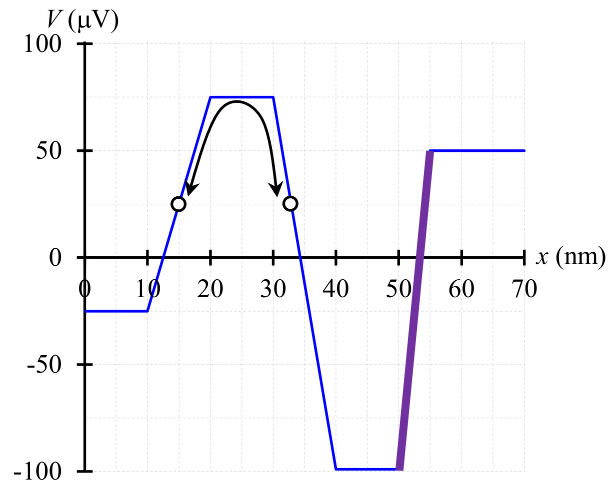
Watch the units and fact that slope is negative!

$$F_x = e\left(\frac{rise}{run}\right)$$

$$F_x = e\left(\frac{150\ \mu\text{V}}{5\ \text{nm}}\right)$$

$$F_x = 1.602 \times 10^{-19}\ \text{C}\left(\frac{150 \times 10^{-6}\ \text{V}}{5 \times 10^{-9}\ \text{m}}\right)$$

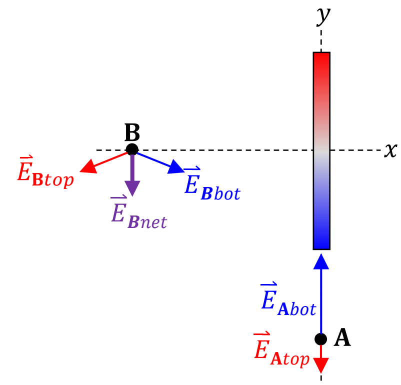
$$\boldsymbol{F_x = +4.81 \times 10^{-15}\ \text{N} = +4.81\ \text{fN}}$$



3b) Suppose the electron was released from rest at $x \approx 33$ nm. Describe the electron's motion in the table below.

Which direction will the electron intially move?	Electrons trend towards areas of *higher* potential. In this case, moving towards higher potential implies moving to a *smaller* value of x. **The electron moves to the *left*.** WATCH OUT! In the bubble analogy we think of the electron as moving along the curved arrow shown on the plot. To make full use of the analogy, we only use the x-component of the motion. The electron never actually moves up or down!
Will the electron reverse direction or not? **If yes,** at what position? **If no,** explain why not?	The electron was released from rest. Initial *kinetic* energy is zero. It speeds up until reaching $x = 30$ nm. Next it travels with constant speed to $x = 20$ nm. The electron slows down as it moves towards $x \approx 15$ nm. When it reaches $x \approx 15$ nm, it has approximately equal *potential* to the starting point. This implies it has equal *potential energy* to the starting point. This implies it must also have equal *kinetic energy* to its starting point ($K = 0$). This implies speed is zero when the electron reaches $x \approx 15$ nm. Furthermore, the slope of the plot implies the electron experiences a force to the right at $x \approx 15$ nm. **The electron therefore reverese direction at $x \approx 15$ nm.**

Because the charge density function has an odd power of y, we expect the top half of the rod to be positively charged and the bottom half of the rod to be negatively charged.



4a) At point **A** all electric field contributions are either upwards or downwards. Since **A** is closer to the negative side of the rod, **net field points upwards at A.**

At point **B** the field contributions form the top & bottom should have equal magnitudes (by symmetry). **Net field points downwards at B.**

4b) **The electric potential at A is negative** since all negative charges are closer than all positive charges.

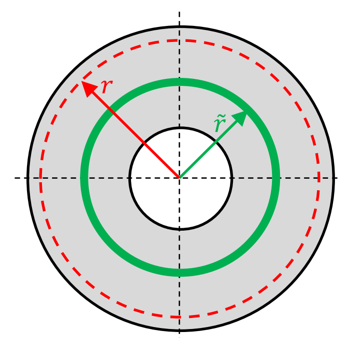
4c) **The electric potential at B is zero.** For every positive charge in the top half of the rod, there is a an equidistant negative charge. Said another way:

For every $k\frac{+dq}{r}$ there is a corresponding $k\frac{-dq}{r}$ with the same radius.

Upon integrating over all contributions the NET potential should be zero.

5a) One finds

$$[\rho] = \frac{[\beta]}{[r^2]} \rightarrow [\beta] = [\rho]\cdot[r^2] = \frac{\mathrm{C}}{\mathrm{m}^3}\cdot\mathrm{m} = \boldsymbol{\frac{\mathrm{C}}{\mathrm{m}}}$$



5b) Use

$$Q_{tot} = \int_R^{3R} \rho\, dV$$

$$Q_{tot} = \int_R^{3R} \frac{\beta}{\tilde{r}^2}\, 2\pi\tilde{r}L\, d\tilde{r}$$

$$Q_{tot} = 2\pi\beta L \int_R^{3R} \frac{d\tilde{r}}{\tilde{r}}$$

$$Q_{tot} = 2\pi\beta L \ln\frac{3R}{R}$$

$$\boldsymbol{Q_{tot} = 2\pi\beta L \ln 3}$$

Notice you can use the units from part a to confirm the units check out. Recall functions have no units.

5c) The Gaussian surface described lies between the inner & outer radius of the object. Only *some* of the total charge is enclosed.
Change the upper limit on the integral to r instead of $3R$.

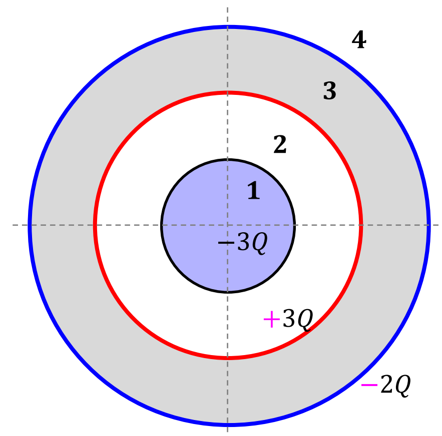
$$EA_{Gaussian} = \frac{q_{in}}{\epsilon_0}$$

$$E(2\pi rL) = \frac{1}{\epsilon_0} 2\pi\beta L \int_R^{r} \frac{d\tilde{r}}{\tilde{r}}$$

$$E(2\pi rL) = \frac{1}{\epsilon_0} 2\pi\beta L \int_R^{r} \frac{d\tilde{r}}{\tilde{r}}$$

$$\boldsymbol{E = \frac{\beta}{\epsilon_0}\cdot\frac{1}{r}\ln\frac{r}{R}}$$

6a) The *inner* surface of the *conducting outer shell* becomes polarized.
The *inner* surface of the *conducting outer shell* carried charge opposite to total charge inside. Therefore *inner surface* of the *outer shell* carries charge $+3Q$.



The *outer* shell (both *inner* & *outer surfaces*) carries total charge Q.
This implies the outer surface of the outer shell carries charge $-2Q$.
Notice $q_{total\ outer\ shell} = q_{inner\ surface} + q_{outer\ surface}$.

Finally, surface charge density at radius $3a$ is given by

$$\sigma_{3a} = \frac{q_{3a}}{A_{3a}} = \frac{-2Q}{4\pi(3a)^2} = -\frac{\boldsymbol{Q}}{\boldsymbol{18\pi a^2}} \approx -\boldsymbol{0.01768}\frac{\boldsymbol{Q}}{\boldsymbol{a^2}}$$

6b)

$$E_1 A_{Gaussian} = \frac{q_{in}}{\epsilon_0}$$

$$E_1 4\pi r^2 = \frac{1}{\epsilon_0}\left(-3Q \cdot \frac{V_{Gaussian}}{V_{total}}\right)$$

$$E_1 4\pi r^2 = \frac{-3Q}{\epsilon_0}\left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a^3}\right)$$

$$E_1 = \frac{-3Q}{4\pi a^3 \epsilon_0} \cdot r$$

$$\boldsymbol{E_1} = \frac{\boldsymbol{-3kQ}}{\boldsymbol{a^3}} \cdot \boldsymbol{r}$$

6c) Outside a sphere we expect the field to be that of a point charge.

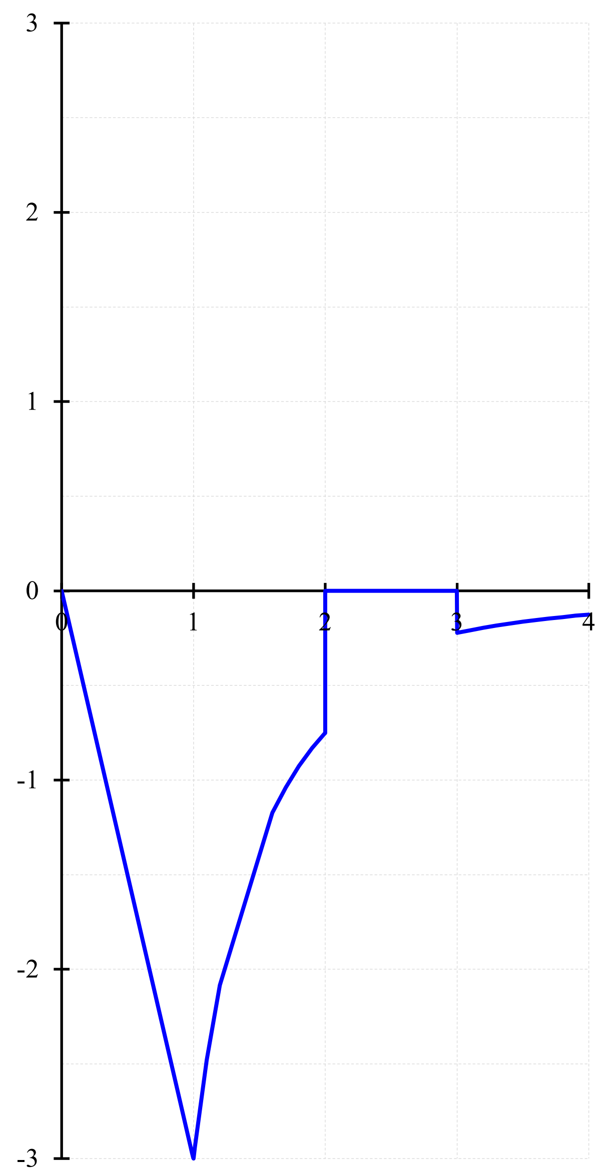
$$\boldsymbol{E_2} = -\frac{\boldsymbol{3kQ}}{\boldsymbol{r^2}}$$

6d) Inside a conductor we expect the (static) field to be **zero**!

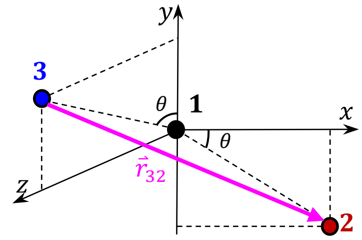
6e) Once again, we are outside of a sphere. Be sure to use NET charge enclosed.

$$\boldsymbol{E_3} = -\frac{\boldsymbol{2kQ}}{\boldsymbol{r^2}}$$

6f) Shape of plot shown at right.
Horizontal axis in units of the given radius a.
Vertical axis in units of $\frac{kQ}{a^2}$.



7a) It probably helps to first find the positions of **2** & **3** in Cartesian form.



$$\vec{r}_2 = 4.00\text{ m}\,(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$$

$$\vec{r}_2 = \left(3.7\underline{0}3\hat{\imath} - 1.5\underline{1}14\hat{\jmath}\right)\text{ m}$$

Out of habit I keep an extra digit when the first digit of a number is 1…

$$\vec{r}_3 = 4.00\text{ m}\left(\cos\theta\,\hat{\jmath} + \sin\theta\,\hat{k}\right)$$

$$\vec{r}_3 = \left(3.7\underline{0}3\hat{\jmath} + 1.5\underline{1}14\hat{k}\right)\text{ m}$$

Now get displacement *from* **3** *to* **2** using final position minus initial position.

$$\vec{r}_{32} = \vec{r}_2 - \vec{r}_3$$

$$\vec{r}_{32} = \left(3.7\underline{0}3\hat{\imath} - 5.2\underline{1}5\hat{\jmath} - 1.5\underline{1}14\hat{k}\right)\text{ m}$$

One finds the magnitude of this vector is $r_{32} = 6.5\underline{7}2\text{ m}$.

We are asked to find the direction given by

$$\boldsymbol{\hat{r}_{32} = \frac{\vec{r}_{32}}{r_{32}} = 0.563\hat{\imath} - 0.794\hat{\jmath} - 0.230\hat{k}}$$

Notice the units drop out as expected when we compute a unit vector. Don't you miss Python right about now???

7b) Want the net Coulomb force **on 1**.

$$\vec{F}_{\text{NET on 1}} = \vec{F}_{\text{2 on 1}} + \vec{F}_{\text{3 on 1}}$$

In this case, both forces have equal magnitudes and it is quick to get components using SOH CAH TOA.

One should get the same final result if you use $\frac{kq\vec{r}}{r^3}$ or $\frac{kq\hat{r}}{r^2}$ but it probably takes longer.

$$\vec{F}_{\text{2 on 1}} = \frac{k\,|100\ \mu\text{C}|\,|300\ \mu\text{C}|}{(4.00\text{ m})^2}(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

$$\vec{F}_{\text{2 on 1}} = 16.586\frac{\text{N}}{\text{C}}(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

$$\vec{F}_{\text{2 on 1}} = \left(-15.\underline{6}07\hat{\imath} + 6.3\underline{6}9\hat{\jmath}\right)\frac{\text{N}}{\text{C}}$$

Verify the direction of the force *on* q_1 is in the direction expected (*away* from like-signed charge q_2).

$$\vec{F}_{\text{3 on 1}} = \frac{k\,|100\ \mu\text{C}|\,|-300\ \mu\text{C}|}{(4.00\text{ m})^2}\left(\cos\theta\,\hat{\jmath} + \sin\theta\,\hat{k}\right)$$

$$\vec{F}_{\text{3 on 1}} = 16.586\frac{\text{N}}{\text{C}}\left(-\cos\theta\,\hat{\jmath} + \sin\theta\,\hat{k}\right)$$

$$\vec{F}_{\text{3 on 1}} = \left(15.\underline{6}07\hat{\jmath} + 6.3\underline{6}9\hat{k}\right)\frac{\text{N}}{\text{C}}$$

Verify the direction of the force *on* q_1 is in the direction expected (*towards* opposite-signed charge q_3).

$$\vec{F}_{\text{NET on 1}} = \left(-15.\underline{6}07\hat{\imath} + 6.3\underline{6}9\hat{\jmath}\right)\frac{\text{N}}{\text{C}} + \left(15.\underline{6}07\hat{\jmath} + 6.3\underline{6}9\hat{k}\right)\frac{\text{N}}{\text{C}}$$

$$\vec{F}_{\text{NET on 1}} = \left(-15.\underline{6}07\hat{\imath} + 21.\underline{9}8\hat{\jmath} + 6.3\underline{6}9\hat{k}\right)\frac{\text{N}}{\text{C}}$$

Don't forget, I asked for the *magnitude* of this force.

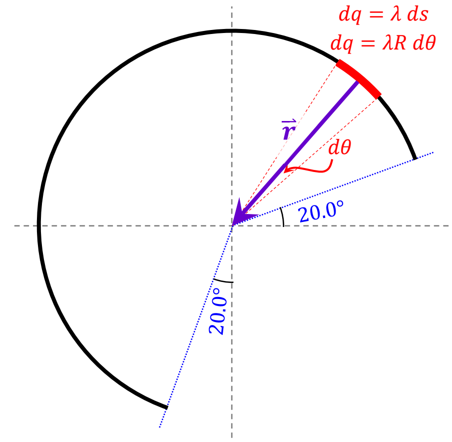
Paying attention to these subtle details is exactly the kind of stuff you encounter in real-life engineering work.

$$\boldsymbol{F_{\text{NET on 1}} = 27.7\frac{\text{N}}{\text{C}}}$$

Side note: a common alternative unit is $\frac{\text{V}}{\text{m}}$.

8a) Even though Q is unknown, pretend it is known then solve for it in the end.

$$\vec{E}_{total} = \int_i^f d\vec{E} = \int_i^f \frac{k\, dq\, \vec{r}}{r^3} = \int_i^f \frac{k\, \lambda\, ds\, \vec{r}}{r^3}$$



Pay close attention to the subtle difference between ds & $d\theta$.

Think: λ has units of $\frac{\text{C}}{\text{m}}$.

Multiplying by $d\theta$ dose NOT give the correct units of C for dq!!!

Since the rod is uniform we can determine

$$\lambda = \frac{Q_{total}}{total\ arclength} = \frac{Q_{total}}{R\theta_{total}} = \frac{Q}{R(230°)} = \frac{Q}{R(4.014\text{ rad})} = \mathbf{0.2491}\frac{\boldsymbol{Q}}{\boldsymbol{R}}$$

WATCH OUT! When doing calculus we must use *radians*!!!

While we're converting, notice $\theta_{min} = 0.3491\text{ rad}$ & $\theta_{max} = 4.363\text{ rad}$.

Notice the *uniform* density can pull out of the integral!

Also, watch out for the r-vector pointing from the source (the arc) to the POI (the origin).

$$\vec{E}_{total} = k\lambda \int_{\theta_{min}}^{\theta_{max}} \frac{R\, d\theta\, (-R\cos\theta\, \hat{\imath} - R\sin\theta\, \hat{\jmath})}{R^3}$$

$$\vec{E}_{total} = \frac{k\left(0.2491\frac{Q}{R}\right)}{R} \int_{\theta_{min}}^{\theta_{max}} (-\cos\theta\, \hat{\imath} - \sin\theta\, \hat{\jmath})\, d\theta$$

Now notice that this charge configuration should produce equal contributions *down* and *to the right* by symmetry. We only need to do one of the integrals! I'll choose to do the first one for no good reason…

$$E_x = -0.2491\frac{kQ}{R^2}\int_{\theta_{min}}^{\theta_{max}} (-\cos\theta)\, d\theta$$

$$E_x = -0.2491\frac{kQ}{R^2}\Big[\sin\theta\Big]_{\theta_{min}}^{\theta_{max}} \approx 0.3193\frac{kQ}{R^2}$$

Think: we expect this result to be smaller than $\frac{kQ}{R^2}$ since all charge is distance R for the POI but we also expect some cancellation due to contributions produced by source charges on the left and right sides of the POI. Looks good.

Now recall the symmetry argument (same components *to the right* and *down* at the origin) to get

$$\vec{E}_{total} = 0.3193\frac{kQ}{R^2}(\hat{\imath} - \hat{\jmath})$$

Finally, get the magnitude of the result so you can solve for Q.

$$E_{total} \approx 0.45\underline{1}5\frac{kQ}{R^2}$$

$$\boldsymbol{Q \approx 2.21\frac{ER^2}{k} = \left(2.46\times 10^{-10}\frac{C^2}{N\cdot m}\right)ER^2}$$

NOTICE: You only need units on the number if you plug in the value of k (which has units).

8b) Answer is $E' < E$.

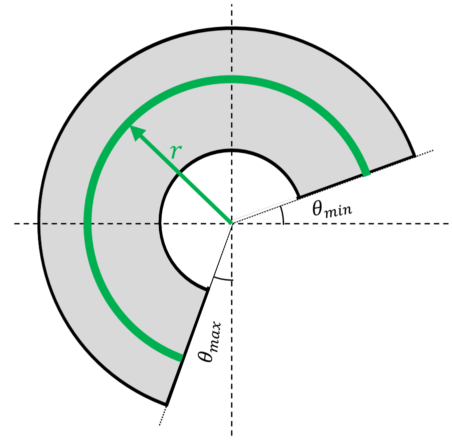
In the new charge configuration, we expect slightly more cancellations from the charge contributions.

The new charges added will cancel more contributions from the already existing arc.

If you care, I did the math and found

$$E' \approx 0.300\frac{kQ'}{R^2} = 0.300\frac{k\lambda\theta'_{total}}{R^2} = 0.300\frac{k\lambda(1.1739\theta_{total})}{R^2} = 0.352\frac{kQ}{R^2} < E$$

Side note: if you had used the same *charge* (instead of same *charge density*) the field would be even weaker!



The extra credit is actually doable but attention to detail is brutal here.
We first need to shift the previous result from R & Q to r & dq:

$$E \approx 0.45\underline{1}5\frac{kQ}{R^2} \quad \rightarrow \quad dE = 0.45\underline{1}5\frac{k\,dq}{r^2}$$

Since all field contributions at the origin point the same direction, this is one of the *rare* cases we can integrate the magnitudes of each contribution to get the net contribution. Normally one must integrate $d\vec{E}$ not dE.

$$E_{total} = \int_R^{3R} 0.45\underline{1}5\frac{k\,dq}{r^2}$$

Here one must recognize we should treat the arc as a 2D object since we are using it to build up a 2D shape.

$$dq = \sigma\, dA = \sigma\, r\theta_{total}\, dr$$

Since the object carries uniform charge contribution

$$\sigma = \frac{Q}{A_{outer} - A_{inner}} = \frac{Q}{0.6389[\pi(3R)^2 - \pi R^2]} = \frac{Q}{0.6389(8\pi R^2)} \approx 0.06228\frac{Q}{R^2}$$

Note: the factor of 0.6389 comes from the fact we only have 230° out of a total 360° giving $\frac{230°}{360°} = 0.6389$.

From there, plug in and crank it out

$$E_{total} = 0.45\underline{1}5k\left(0.06228\frac{Q}{R^2}\theta_{total}\right)\int_R^{3R}\frac{1}{r}\,dr$$

$$E_{total} = 0.11288\frac{kQ}{R^2}[\ln r]_R^{3R}$$

$$\boldsymbol{E_{total} = 0.1240\frac{kQ}{R^2}}$$

If you chose to code it, you would probably want to do something like what is shown on the next page.

Note: if you look carefully at the code you will notice a trick I used.
According to the math above, setting $Q = \frac{1}{8.99\times10^9}$ C & $R = 1$ m should output field magnitude $0.1240\,\frac{\text{N}}{\text{C}}$.

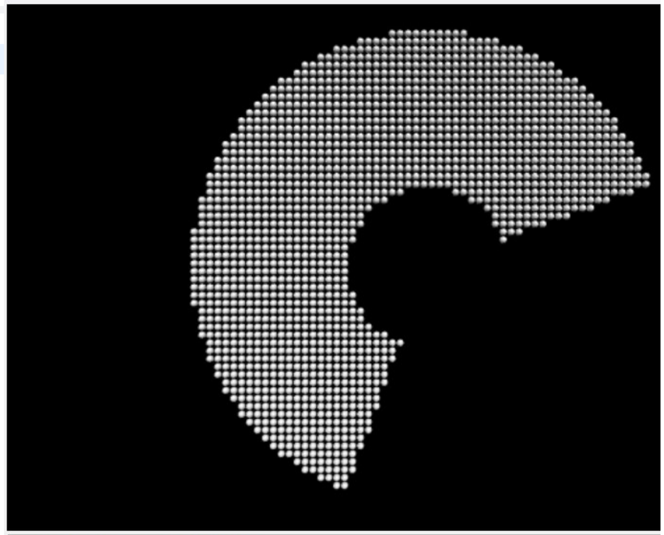
```
Web VPython 3.2

k=8.99e9
Q=1/8.99e9
R_inner = 1
R_outer = 3
theta_min = radians(20)
theta_max = radians(250)
theta_total = theta_max - theta_min

fraction_of_circle = theta_total / ( 2*pi )
A_outer = fraction_of_circle * pi*R_outer**2
A_inner = fraction_of_circle * pi*R_inner**2
A_total = A_outer - A_inner
sigma = Q/A_total

dx = 0.1
dy = dx
POI = vec(0,0,0)
E_total = vec(0,0,0)
for x in arange(-1*R_outer, R_outer, dx):
    for y in arange(-1*R_outer, R_outer, dy):
        my_x = x+0.5*dx
        my_y = y+0.5*dy
        my_pos = vec(my_x, my_y, 0)
        my_angle = atan2(my_y, my_x)
        condition1 = R_inner < my_pos.mag < R_outer
        condition2 = theta_min < my_angle < theta_max
        condition3 = 0 < (pi+my_angle) < (theta_max-pi)
        if condition1 and (condition2 or condition3):
            ball=sphere(pos=my_pos,
                        radius=0.45*dx,
                        dq = sigma*dx*dy)
            r= POI-ball.pos
            E_total+=k*ball.dq*r/r.mag**3

print(f"E_total.mag = {E_total.mag:.4f} N/C")
```



E_total.mag = 0.1234 N/C
% diff = -0.4%