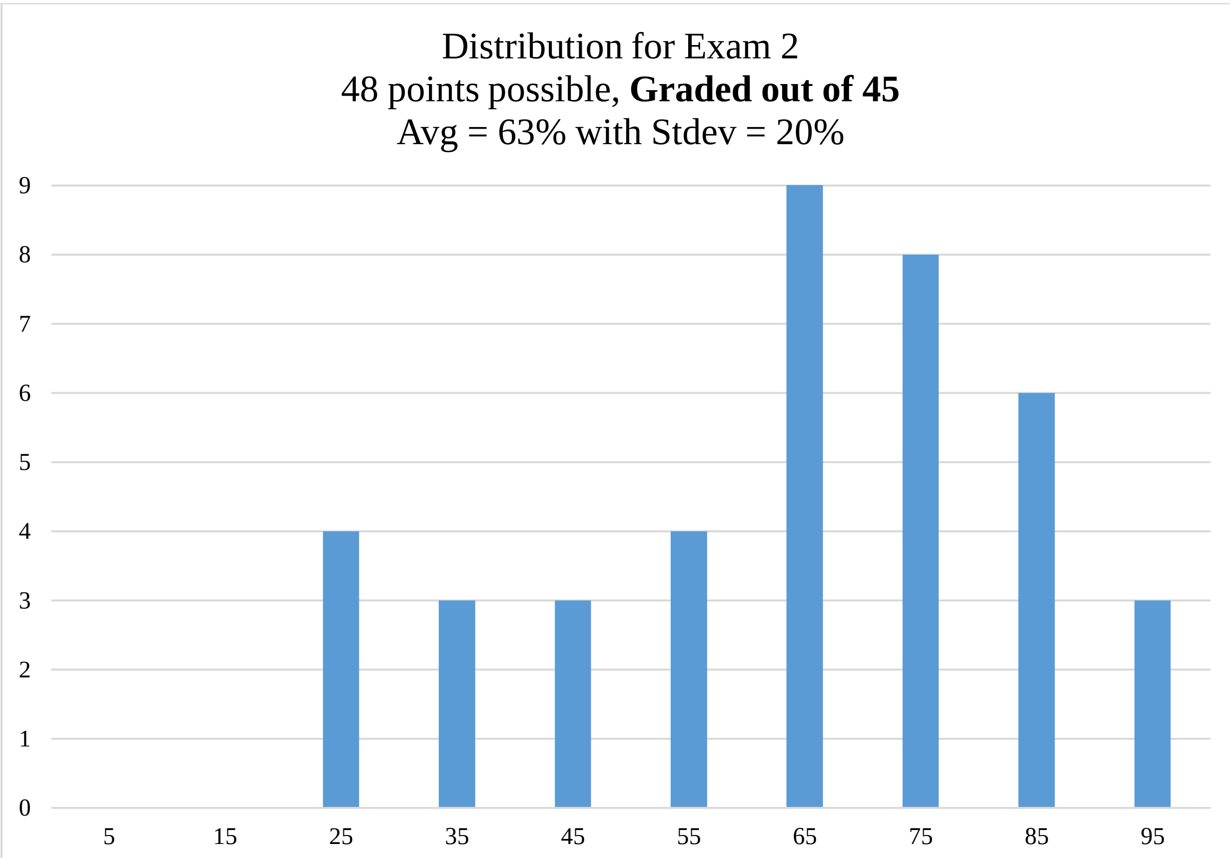
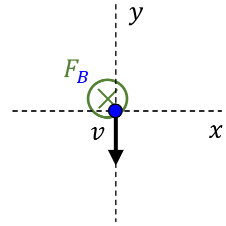
163fa23t2aSoln

Distribution on this page.
Solutions begin on the next page.



1a) We know

$$\vec{F}_B = q\vec{v} \times \vec{B}_{ext}$$



For this special case, let's plug in what we know for $\vec{F}, q, \&\ \vec{v}$ and put in a general guess for $\vec{B}_{ext}$.

$$\vec{F}_B = (-e)(-v\hat{\jmath}) \times \left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)$$

$$-F_B\hat{k} = (ev\hat{\jmath}) \times \left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)$$

From this one can determine the following:

- B_x must be positive (to ensure the $\hat{\imath}$ component of the cross-product turns out negative…$\hat{\jmath} \times \hat{\imath} = -\hat{k}$).
- B_y could be anything as that component of the cross-product will drop out (circle impossible to determine)
- B_z must be zero (otherwise the force would have an $\hat{\imath}$-component…and we were told it doesn't have one)

1b) To counteract the deflection $-F_B\hat{k}$, we'd require an electric force *out of the page*.
It is possible, but none of the drawn plates could cause such a deflection.
We would require parallel plates which lie parallel to the xy-plane.
The positive plate would be in front of the electron (positive z-coordinate) while the negative plate would lie behind the electron (negative z-coordinate).

2a) The capacitors were placed in series and then allowed to reach full charge.
When this occurs, we know the charges on the capacitors must be identical.
Thus, it behooves us to write the energy equation in terms of charge.

$$U = \frac{1}{2}Q\Delta V = \frac{1}{2}Q\left(\frac{Q}{C}\right) = \frac{Q^2}{2C}$$

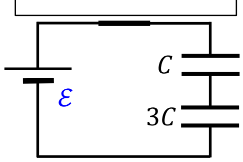
We see that for two capacitors with equal charge, **the *smaller* capacitor has more energy ($U_C > U_{3C}$).**

2b) In the final reconnected circuit, equilibrium is reached when the two capacitors have equal potential difference.
The total stored charge from both capacitors will move around until this state is reached.
Final charge on each capacitor and stored energy in each capacitor are not equal.

2c) You could do the math and get the results shown below.
Alternatively, you could follow this logic:

- Want caps to have same final voltage
- Require $\Delta V_n = \frac{Q_n}{C_n}$ (here n is the capacitor number…either 1 or 3)
- Implies larger cap requires more charge
- Implies Q_C decreases while Q_{3C} increases
- Implies $\Delta V_1' < \Delta V_1$ while $\Delta V_3' > \Delta V_3$!!!

Charging circuit
with caps
fully charged



$\Delta V_1 = \frac{3}{4}\mathcal{E}$

$Q_1 = \frac{3}{4}\mathcal{E}C$

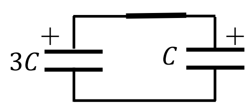
$\Delta V_2 = \frac{1}{4}\mathcal{E}$

$Q_2 = \frac{3}{4}\mathcal{E}C$

Reconnected circuit
after equilibrium
reached

$\Delta V_3{}' = \frac{3}{8}\mathcal{E}$

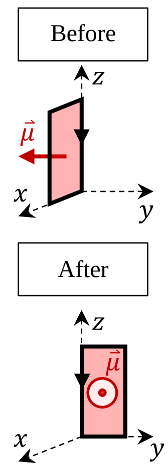
$Q_3{}' = \frac{9}{8}\mathcal{E}C$



$\Delta V_1{}' = \frac{3}{8}\mathcal{E}$

$Q_1{}' = \frac{3}{8}\mathcal{E}C$

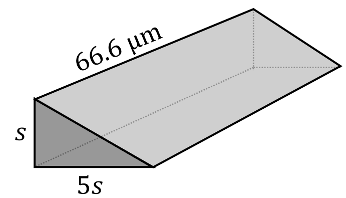
PAY CLOSE ATTENTION TO THE COORDINATE SYSTEM!!!



3a) The magnetic moment is found by curling your right hand in the direction of current flow around the loop. I found $\hat{\mu} = -\hat{j}$.

3b) We know the loop will tend to align its magnetic moment with the external field.
An external magnetic field directed into the page ($\hat{B}_{ext} = +\hat{\imath}$) should cause the desired rotation.
Take a moment to verify no other orientation of $\vec{B}_{ext}$ could produce the desired rotation.

4a) Use



$$R = \frac{\rho L}{A}$$

$$R = \frac{\rho_0(1 + \alpha\Delta T)L}{\frac{1}{2}s(5s)}$$

$$s = \sqrt{\frac{2\rho_0(1 + \alpha\Delta T)L}{5R}}$$

Be careful plugging in parameters from the equations sheet. In particular, $\Delta T = 144°\text{C} - 20.0°\text{C} = 124°\text{C}$.
Also, don't forget to use the table on the equation sheet to get ρ_0 & α for carbon.
You are expected to know the units for the parameters in this table.

$$s = \sqrt{\frac{2(3.5 \times 10^{-5}\ \Omega \cdot \text{m})\left[1 + \left(-0.5 \times 10^{-3}\frac{1}{°\text{C}}\right)(124°\text{C})\right](66.6 \times 10^{-6}\ \text{m})}{5(333 \times 10^{-9}\ \Omega)}}$$

$$\boldsymbol{s = 51.2\ \text{mm}}$$

Notice the problem statement indicated figure not to scale!

4b) In this problem, we are treating carbon is a semi-conductor (notice $\alpha < 0$).
The warmer carbon gets, the lower its resistance.

4c) Drilling the hole would decrease the cross-sectional area and increase resistance.

5a) I redrew the circuit as shown at right.
Notice the inclusion of the black and white nodes helps keep things correct.
First combine $7C$ & C in parallel.

$$C_{17} = 7C + C = 8C$$

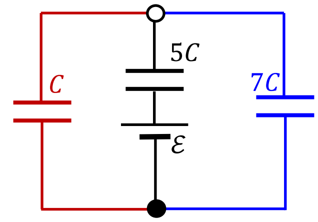
Next combine C_{17} & $5C$ in series.

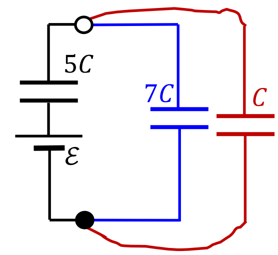
$$C_{157} = \frac{C_{17}(5C)}{C_{17} + 5C}$$

$$\boldsymbol{C_{157} = \frac{40}{13}C}$$

$$\boldsymbol{C_{157} = 3.0\underline{7}7C}$$

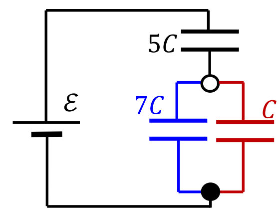
Note: in this problem it *is* acceptable to write your final answer as a fraction since we are assuming given parameters have *infinite* sig figs.

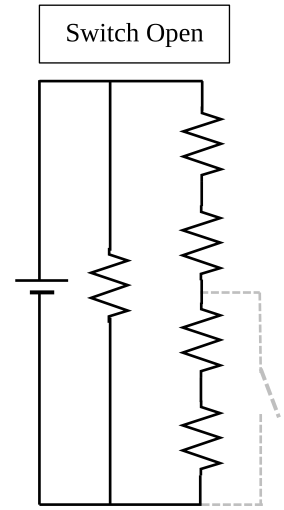




5b) We know C_{17} & $5C$ share the same charge.
Furthermore, $7C$ & C split up the charge associated with C_{17}.
From this we immediately know Q_5 is the largest.
Since $7C$ & C are in parallel, they have the same potential difference.
Since $Q = \Delta V C$, the larger of these two capacitors must carry more of the charge!

$$\boldsymbol{Q_5 > Q_7 > Q_1}$$





6a) First consider the upper circuit shown at right.
When the switch is open, that branch has no effect on the circuit.
Assume each individual resistor has resistance R.
Right branch has *four* resistors in series giving that branch resistance $4R$.
Total resistance is thus

$$R_{eq} = \frac{R \cdot 4R}{R + 4R} = \frac{4}{5}R$$

Think: the middle branch has resistance R.
Adding the right branch gives more paths for current; thus less resistance than R.

Now use Ohm's law with the equivalent resistance.

$$\Delta V_{battery} = i_{battery} R_{eq}$$

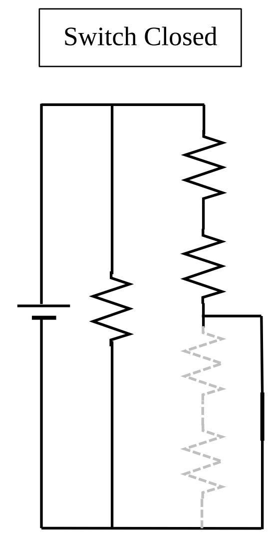
$$R_{eq} = \frac{\Delta V_{battery}}{i_{battery}}$$

$$\frac{4}{5}R = \frac{\Delta V_{battery}}{i_{battery}}$$

$$R = \frac{5}{4} \cdot \frac{6.00\ \mathrm{V}}{1.00\ \mathrm{A}}$$

$$\boldsymbol{R = 7.50\ \Omega}$$

Note: in this problem it *IS NOT* acceptable to write your final answer as a fraction since we are assuming given parameters have *finite* sig figs. Also, our standard is 3 sig figs on final results.



6b) With the switch closed the bottom two R's are effectively removed from the right branch.

$$R'_{eq} = \frac{R \cdot 2R}{R + 2R} = \frac{2}{3}R$$

Since we change resistance but not the battery voltage, we expect current to change.
It seems prudent to rewrite power in terms of resistance and voltage!

$$\mathcal{P}'_{battery} = f\mathcal{P}_{battery}$$

$$f = \frac{\mathcal{P}'_{battery}}{\mathcal{P}_{battery}}$$

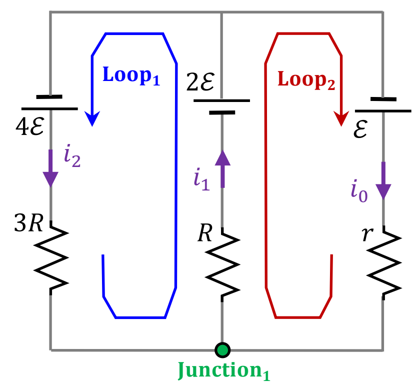
$$f = \frac{\left(\frac{\Delta V^2_{battery}}{R'_{eq}}\right)}{\left(\frac{\Delta V^2_{battery}}{R_{eq}}\right)}$$

$$f = \frac{R_{eq}}{R'_{eq}}$$

$$f = \frac{\frac{4}{5}R}{\frac{2}{3}R}$$

$$\boldsymbol{f = \frac{6}{5} = 1.200}$$

Think: with less total resistance, it should be easier for the battery to push current and thus deliver more power.
It makes sense to have a factor larger than 1. **Note:** I asked for a number with three sig figs.



In this circuit, it is hopefully obvious current should flow upwards in the middle branch and downwards on the two other branches.
Of course, if you goofed on that the math should still work out.

$$\textbf{Loop}_1: -i_1R + 2\mathcal{E} + 4\mathcal{E} - i_2(3R) = 0 \quad \rightarrow \quad -i_1R + 6\mathcal{E} - i_2(3R) = 0$$

$$\textbf{Loop}_2: -i_1R + 2\mathcal{E} + \mathcal{E} - i_0(r) = 0 \quad \rightarrow -i_1R + 3\mathcal{E} - i_0r = 0$$

$$\textbf{Junction}_1: i_0 + i_2 = i_1$$

WATCH OUT! In this problem i_0 is known while r is unknown.
For no good reason, I choose to first eliminate i_1 in both Loop equations.

$$\textbf{Loop}_1: -(i_0 + i_2)R + 6\mathcal{E} - i_2(3R) = 0 \quad \rightarrow \quad -i_0R + 6\mathcal{E} - i_2(4R) = 0$$

$$\textbf{Loop}_2: -(i_0 + i_2)R + 3\mathcal{E} - i_0r = 0 \quad \rightarrow \quad -i_0R - i_2R + 3\mathcal{E} - i_0r = 0$$

To me it seemed easiest to multiply the bottom equation by -4 then add it to the upper equation.

$$\textbf{Loop}_1: \quad -i_0R + 6\mathcal{E} - i_2(4R) = 0$$

$$\underline{(-4) \times \textbf{Loop}_2: \quad 4i_0R + 4i_2R - 12\mathcal{E} + 4i_0r = 0}$$

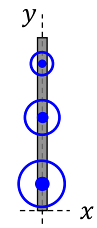
$$3i_0R - 6\mathcal{E} + 4i_0r = 0$$

Now solve for r as requested

$$r = \frac{6\mathcal{E} - 3i_0R}{4i_0}$$

Seems reasonable to clean this up a bit

$$\boldsymbol{r = \frac{3}{2} \cdot \frac{\mathcal{E}}{i_0} - \frac{3}{4}R}$$



8a) Use the right hand rule for the equation $\vec{F} = \int i\, d\vec{s} \times \vec{B}_{ext}$.
Unfortunately, this problem statement never specified a current direction.
We know the force would be $+\hat{\imath}$ for upwards current or $-\hat{\imath}$ for downwards current.
Best answer is impossible to determine without more info.

8b) Check the units using our standard method.

$$[\vec{B}] = \frac{[\beta]}{[(y^2+d^2)^{1/2}]}[\hat{k}]$$

$$[\beta] = \frac{[\vec{B}][(y^2+d^2)^{1/2}]}{[\hat{k}]}$$

$$[\beta] = \frac{\mathrm{T}\cdot\mathrm{m}}{[\text{no units}]}$$

$$\boldsymbol{[\beta] = \mathrm{T}\cdot\mathrm{m}}$$

8c) Since we are only getting the magnitude, it is fine to assume upwards current (since direction will drop out).

$$\vec{F} = \int_0^{2d} i\, d\vec{s} \times \vec{B}_{ext}$$

$$\vec{F} = \int_0^{2d} i\,(dy\,\hat{\jmath}) \times \left(\frac{\beta}{(y^2+d^2)^{1/2}}\hat{k}\right)$$

$$\vec{F} = \beta i \int_0^{2d} (\hat{\jmath}\times\hat{k}) \left(\frac{dy}{(y^2+d^2)^{1/2}}\right)$$

$$\vec{F} = \beta i \int_0^{2d} (\hat{\imath}) \left(\frac{dy}{(y^2+d^2)^{1/2}}\right)$$

$$\vec{F} = \beta i(\hat{\imath}) \int_0^{2d} \frac{dy}{(y^2+d^2)^{1/2}}$$

Use the integral table on the exam equation sheet by setting $a = d$

$$\vec{F} = \beta i(\hat{\imath}) \left[\ln\left|y + \sqrt{y^2+d^2}\right|\right]_0^{2d}$$

WATCH OUT! Always check if the zero limit drops out or if it produces a non-zero term.

$$\vec{F} = \beta i(\hat{\imath}) \left[\ln\left|2d + \sqrt{(2d)^2 + d^2}\right| - \ln\left|0 + \sqrt{0^2+d^2}\right|\right]$$

Common error was forgetting to square the 2. Simplifying this result shows respect to your reader.

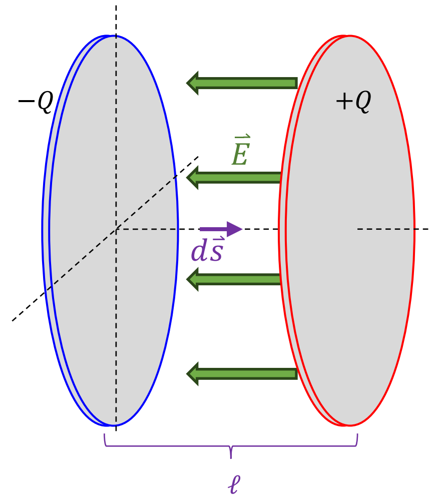
$$\vec{F} = \beta i(\hat{\imath})\left[\ \ln\left|2d + \sqrt{5}d\right| - \ln|d|\ \right] = \ln\left|\frac{2d+\sqrt{5}d}{d}\right| \beta i(\hat{\imath}) = \ln\left|2+\sqrt{5}\right| \beta i(\hat{\imath})$$

We were asked to determine the magnitude.

$$\boldsymbol{F = \ln\left|2+\sqrt{5}\right| \beta i}$$

$$\boldsymbol{F \approx 1.444\beta i}$$

Think: we know the units of β from part b. Use this to verify the units of your final result make sense.
By comparing to the equation $F = iLB$ we see units of N are equivalent to $\mathrm{A}\cdot\mathrm{m}\cdot\mathrm{T}$.



9) When deriving capacitance, do the following:

- Get $\vec{E}$ using either a memorized result or from Gauss's Law
- Compute $\Delta V = -\int_i^f \vec{E} \cdot d\vec{s}$
- Solve for the ratio $C = \frac{Q}{\Delta V}$

From Gauss's Law (or the equation sheet) we know

$$\vec{E} = \frac{\sigma}{\epsilon_0}(-\hat{\imath})$$

Here I know the direction since we expect the field to point away from the positive plate and towards the negative plate.

To be clear, σ in this derivation comes from the surface charge density of a single plate. We should rewrite this result using

$$\sigma = \frac{Q}{A} = \frac{Q}{\pi R^2}$$

This gives

$$\vec{E} = \frac{Q}{\epsilon_0 \pi R^2}(-\hat{\imath})$$

Now use

$$\Delta V = -\int_i^f \vec{E} \cdot d\vec{s}$$

$$\Delta V = -\int_0^\ell \left(\frac{Q}{\epsilon_0 \pi R^2}(-\hat{\imath})\right) \cdot dx\,\hat{\imath}$$

Notice the minus signs cancel.
Notice $\hat{\imath} \cdot \hat{\imath} = 1$.
Notice the limits go from the starting plate at the origin to the other plate.

$$\Delta V = \int_0^\ell \frac{Q}{\epsilon_0 \pi R^2} dx$$

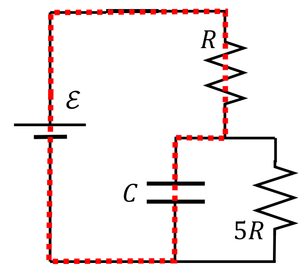
$$\Delta V = \frac{Q}{\epsilon_0 \pi R^2} \ell$$

Now solve for the ratio to find

$$C = \frac{Q}{\Delta V}$$

$$\boldsymbol{C = \frac{\epsilon_0 \pi R^2}{\ell}}$$

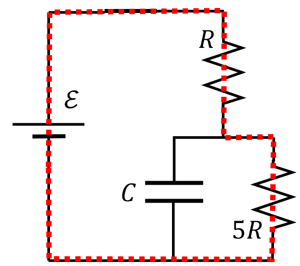
Note: it is customary to write final answers in terms of the parameters given in the problem statement.



10a) When the switch is first closed, the capacitor is uncharged (acts like a *short*).
Current takes the path shown by the red dotted line in the figure at right.

Alternatively, we could say it preserves its voltage of zero.
Since it is in parallel with $5R$, the voltage across $5R$ must also be zero.
Since voltage across $5R$ is zero, no current is flowing through $5R$.

$$i = \frac{\mathcal{E}}{R}$$



10b) After a long time, the capacitor is fully charged (acts like a *break*).
Current takes the path shown by the red dotted line in the figure at right.
Notice resistors R & $5R$ are in series.

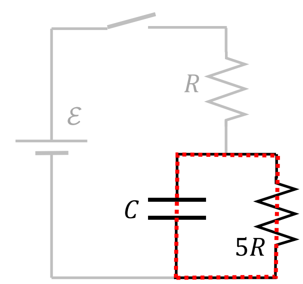
$$i = \frac{\mathcal{E}}{6R}$$

10c) The capacitor is in parallel with $5R$.
Therefore voltage across the capacitor must equal voltage across $5R$.

$$\Delta V_C = \Delta V_{5R}$$

$$\Delta V_C = i_{5R} 5R$$

$$\Delta V_C = \frac{5}{6}\mathcal{E}$$



10d) Once the switch is re-opened, current only flows in the loop containing $5R$ & C.
We are asked about the energy as a function of time.
We know the voltage across the capacitor decays to zero eventually.
Together, these facts make me think I should write an expression for voltage as a function of time and plug it into the equation for energy to get energy as a function of time.

$$\Delta V_C(t) = \Delta V_{C\,max} e^{-t/\tau}$$

Here $\Delta V_{C\,max} = \frac{5}{6}\mathcal{E}$ but I won't bother to plug it in…I think you'll see why.

$$U_C(t) = \frac{1}{2}[\Delta V_C(t)]^2 C$$

$$U_C(t) = \frac{1}{2}\left[\Delta V_{C\,max} e^{-t/\tau}\right]^2 C$$

$$U_C(t) = \frac{\Delta V_{C\,max}^2 C}{2} e^{-2t/\tau}$$

Notice this could be rewritten as

$$U_C(t) = U_{max} e^{-2t/\tau}$$

We were told we want the time when energy has decreased by 33.3% from its max value.

$$U_C(t) = (1 - 33.3\%)U_{max} = 0.667 U_{max}$$

Plugging in gives

$$0.667 U_{max} = U_{max} e^{-2t/\tau}$$

$$0.667 = e^{-2t/\tau}$$

Notice we never actually need to compute U_{max}!!!

$$\ln 0.667 = -\frac{2t}{\tau}$$

$$t = -\frac{\tau}{2}\ln 0.667 = -\frac{5RC}{2}\ln 0.667 = \mathbf{1.012}\boldsymbol{RC}$$

NOTE: notice the minus sign cancels with the $\ln 0.667$ term!
10e) When dielectric is inserted, capacitance doubles. Require half the original resistance to keep same τ.