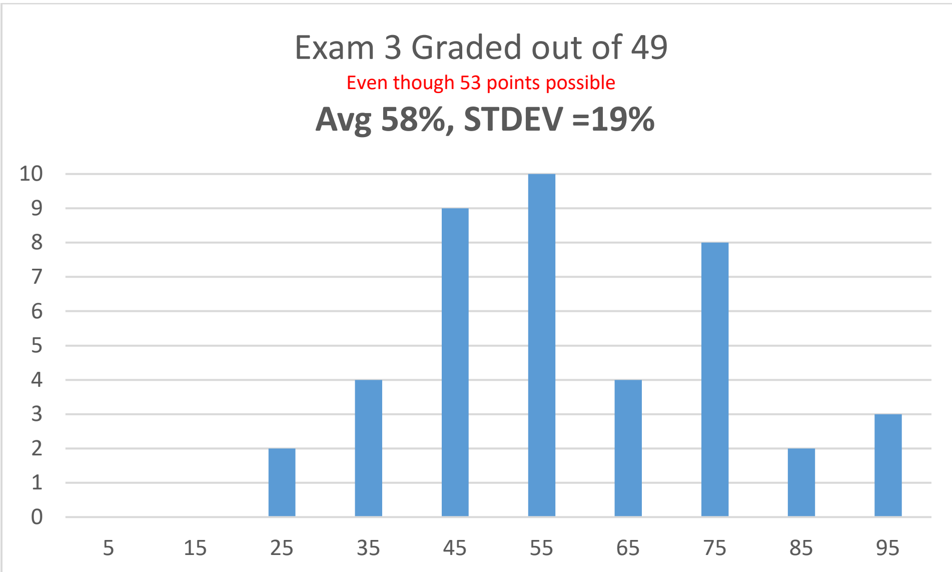
163fa23t3bSoln

Distribution on this page.
Solutions begin on the next page.

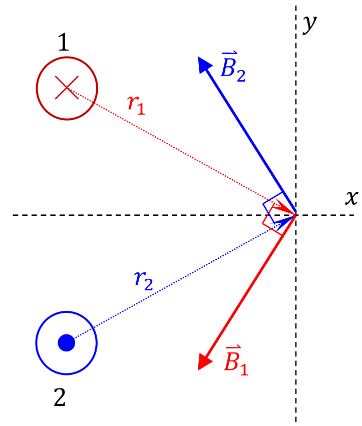


1a) Consider the figure at right.
The wires are equidistant from the origin with equal current.
By symmetry, vertical components of the field cancel (and no field in $\pm\hat{k}$).

$$\boldsymbol{B_x < 0}$$

$$\boldsymbol{B_y = 0}$$

$$\boldsymbol{B_z = 0}$$



1b) The magnitude of field created by each wire is that of an infinitely long straight wire in the directions shown in the top right figure.

$$B_1 = B_2 = \frac{\mu_0 i}{2\pi a}$$

In this case

$$a = r_1 = r_2 = \sqrt{(3.50\text{ cm})^2 + (7.00\text{ cm})^2} \approx 7.826\text{ cm} = 0.07826\text{ m}$$

Here

$$\theta = \tan^{-1}\left(\frac{3.50\text{ cm}}{7.00\text{ cm}}\right) \approx 26.565°$$

Again using the symmetry of the problem:

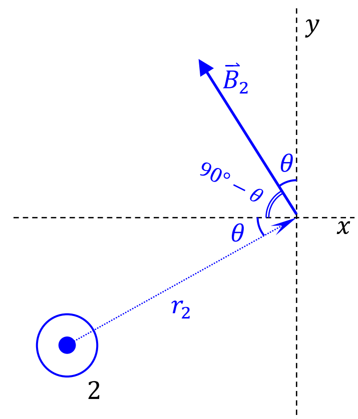
$$B_{total} = 2B_{2x} = 2\frac{\mu_0 i}{2\pi r_2}\sin\theta$$

Rearranging for i gives

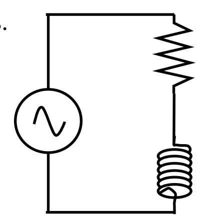
$$i = \frac{\pi r_2 B_{total}}{\mu_0 \sin\theta}$$

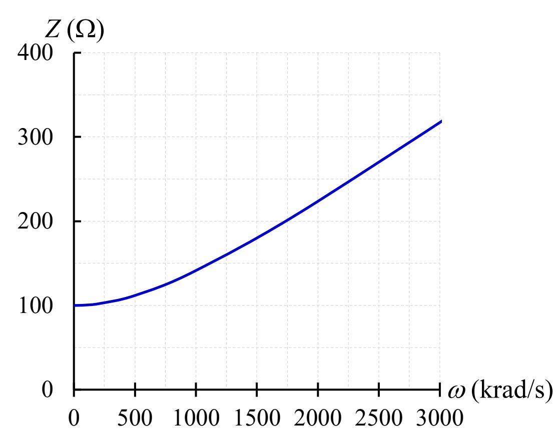
$$i \approx \frac{\pi(0.07826\text{ m})(22.2\times10^{-3}\text{ T})}{\left(4\pi\times10^{-7}\frac{\text{T}\cdot\text{m}}{\text{A}}\right)\sin(26.565°)}$$

$$\boldsymbol{i \approx 9.71\text{ kA}}$$

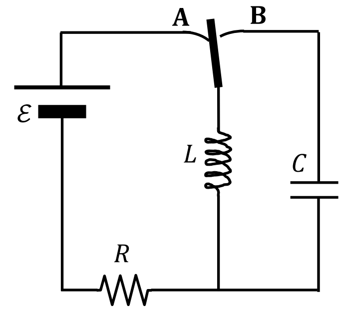


2a) The inductor in following circuit behaves as a short at low frequencies and break at high frequencies.





2b) At extremely low frequencies the circuit is essentially DC (inductor has no effect on circuit). Resistance is $\boldsymbol{R \approx 100\ \Omega}$.



3a) The switch is left in position A for a long time.
In steady state the inductor acts like a short.
Steady state current in the left loop is thus $i = \frac{\mathcal{E}}{R}$.
Magnetic energy in the fully energized inductor is

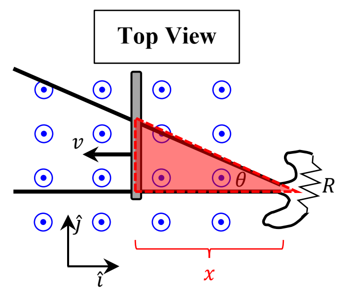
$$U_{Lmax} = \frac{1}{2}Li^2 = \frac{\mathcal{E}^2 L}{2R^2} \approx 1.8904 \times 10^{-5}\text{ J} = \mathbf{35.1\underline{6}\ \mu J}$$

3b) It takes ¼ of a cycle for the first transfer of energy from the inductor to the capacitor.

$$t = \frac{1}{4}\mathbb{T} = \frac{1}{4}\cdot\frac{2\pi}{\omega_0} = \frac{1}{4}\cdot 2\pi\sqrt{LC} \approx \mathbf{38.5\ \mu s}$$

3c) In this idealized circuit, there is no resistor to dissipate energy.
As such, we expect max capacitor energy (stored in electric fields at full charge) should equal max inductor energy (stored in magnetic fields when fully energized).

$$U_{Cmax} = \frac{Q_{max}^2}{2C} = U_{Lmax} \quad \rightarrow \quad Q_{max} = \sqrt{2U_{Lmax}C} \approx \mathbf{4.59\ \mu C}$$



4a) The figure at right shows the triangular current loop.
As the rod moves to the left, loop area increases.
Magnetic flux *out of the page* increases.
Faraday's law tells us *induced EMF opposes the change in flux*.
In this case, induced EMF causes **induced current producing *into the page* flux**.
Current must flow ***upwards*** in the rod.

4b) Use a right hand rule.
Current up in rod, $\vec{B}_{ext}$ directed out of page, force is to the right.
Check: usually we expect magnetic braking (<u>not</u> "magnetic make you go faster") in these types of problems.

4c) The rod is pulled at constant speed but the rails keep getting farther and farther apart.
Area should increase at a faster and faster rate.
Induced EMF (and thus induced current in the rod) should increase.

4d) Do SOH CAH TOA on the triangle to determine height in terms of x & θ.

$$\tan\theta = \frac{height}{x} \quad \rightarrow \quad height = x\tan\theta$$

Loop area is given by

$$A = \frac{1}{2}base \times height = \frac{x^2\tan\theta}{2}$$

Now compute EMF

$$EMF = -\frac{d}{dt}B_{ext}A\cos\theta_{AB}$$

WATCH OUT! In this equation θ_{AB} is not θ!!!
Here θ_{AB} is the angle between the area vector (perpendicular to plane of loop) and $\vec{B}_{ext}$.
In this case $\theta_{AB} = 0$ which implies $\cos\theta_{AB} = 1$.

$$EMF = -\frac{d}{dt}B_{ext}\frac{x^2\tan\theta}{2} = -B_{ext}\frac{\tan\theta}{2}\frac{\boldsymbol{d}}{\boldsymbol{dt}}x^2 = -B_{ext}\frac{\tan\theta}{2}\boldsymbol{2x}\frac{\boldsymbol{dx}}{\boldsymbol{dt}} = -B_{ext}\boldsymbol{xv}\tan\theta$$

We are given current is I. Relate this to EMF (voltage) using Ohm's Law.
Ignore the minus sign in this case. That was essentially used to get current direction in part 4a.

$$EMF = IR = B_{ext}xv\tan\theta \quad \rightarrow \quad \boldsymbol{B_{ext} = \frac{IR}{vx\tan\theta}}$$

Another correct way to determine EMF:

$$\mathcal{E}_{motional} = B_{\perp}\ell v$$

Entire field was $\perp$ to $\vec{v}$ so $B_{\perp} = B$

$$\ell = x\tan\theta$$

$$\mathcal{E} = Bxv\tan\theta$$

5a) **WATCH OUT!** $f_{resonance} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = 8.02\text{ kHz}$ while $f_{operating} = \frac{1}{\mathbb{T}} \approx 18.02\text{ kHz}$

5b) We'll need $\omega_{operating} = 2\pi f_{operating} = 1.1321 \times 10^5\,\frac{\text{rad}}{\text{s}}$.
TIP: Use the *unrounded* value of $f_{operating}$ to avoid intermediate rounding error!

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$Z = \sqrt{R^2 + \left(\omega_{operating}L - \frac{1}{\omega_{operating}C}\right)^2}$$

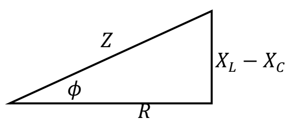
$$Z = \sqrt{(33.3\ \Omega)^2 + \left(\left(1.1321 \times 10^5\,\frac{\text{rad}}{\text{s}}\right)(888 \times 10^{-6}\text{ H}) - \frac{1}{\left(1.1321 \times 10^5\,\frac{\text{rad}}{\text{s}}\right)(444 \times 10^{-9}\text{ F})}\right)^2}$$

$$\boldsymbol{Z = 87.\underline{2}4\ \Omega}$$

5c) Use the phase angle triangle.

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

$$\boldsymbol{\phi \approx +67.\underline{5}6°}$$



5d) $\mathcal{P}_{avg} = i_{RMS}^2 R$. **Most common errors:** $Z \neq R$ when $\omega_{operating} \neq \omega_0$, forgetting Max to RMS conversion.

$$i_{RMS} = \frac{V_{source\ RMS}}{Z}$$

$$i_{RMS} = \frac{\frac{V_{source\ Amplitude}}{\sqrt{2}}}{Z}$$

$$i_{RMS} = \frac{V_{source\ Amplitude}}{\sqrt{2}Z}$$

Plug into $\mathcal{P}_{avg} = i_{RMS}^2 R$:

$$\mathcal{P}_{avg} = \left(\frac{V_{source\ Amplitude}}{\sqrt{2}Z}\right)^2 R$$

$$\mathcal{P}_{avg} = \frac{V_{source\ Amplitude}^2 R}{2Z^2}$$

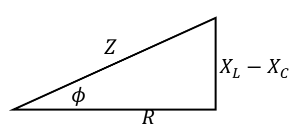
$$\mathcal{P}_{avg} = \frac{(22.2\text{ V})^2(33.3\ \Omega)}{2(87.24\ \Omega)^2}$$

$$\boldsymbol{\mathcal{P}_{avg} \approx 1.078\text{ W}}$$

Alternative style:

$$\mathcal{P}_{avg} = I_{RMS}V_{source\ RMS}\cos\phi$$

$$\mathcal{P}_{avg} = \left(\frac{V_{source\ Amplitude}}{\sqrt{2}Z}\right)\left(\frac{V_{source\ Amplitude}}{\sqrt{2}}\right)\left(\frac{R}{Z}\right)$$

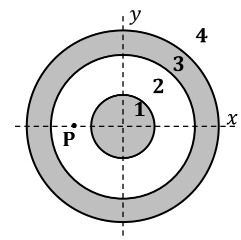


5e) Positive phase angle implies $X_L > X_C$. Inductor is dominating the circuit. Use ELI from ELI the ICE freak. **Current lags source voltage** (I comes after E in ELI). Note: you could also say voltage leads current if you want.

5f) If C decreases $X_C = \frac{1}{\omega C}$ increases. Notice $X_L - X_C$ is more negative. **Phase angle becomes more negative.**

6a) $[k] = \frac{[J]}{[r]^8} = \frac{\mathbf{A}}{\mathbf{m^{10}}}$

6b) For non-uniform density



$$I_{total} = \int_{R_i}^{R_f} J\, dA$$

$$I = \int_0^R k\tilde{r}^8\, 2\pi\tilde{r}\, d\tilde{r}$$

$$I = 2\pi k \int_0^R \tilde{r}^9\, d\tilde{r}$$

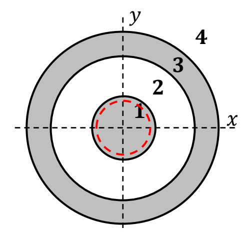
$$I = 2\pi k \left[\frac{\tilde{r}^{10}}{10}\right]_0^R$$

$$I = \frac{\pi k R^{10}}{5}$$

$$\boldsymbol{k = \frac{5I}{\pi R^{10}}}$$

6c) By symmetry, outer shell has no effect on field direction at **P**.
Current for inner shell is directed *into the page*.
Right hand rule gives magnetic field direction at **P** *upwards* $(+\hat{\jmath})$.

6d) Use Ampere's law using the Amperian loop in region 1 (shown at right).



$$B_1 s = \mu_0 I_{enclosed}$$

$$B_1(2\pi r) = \mu_0 \int_0^r k\tilde{r}^8\, 2\pi\tilde{r}\, d\tilde{r}$$

$$B_1(2\pi r) = \mu_0 \frac{\pi k r^{10}}{5}$$

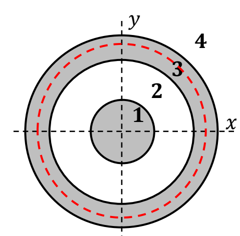
$$B_1(2\pi r) = \mu_0 \frac{\pi\left(\frac{5I}{\pi R^{10}}\right) r^{10}}{5}$$

$$\boldsymbol{B_1 = \frac{\mu_0 I r^9}{2\pi R^{10}}}$$

6e) In region 2 we are completely outside the wire, enclosing all current from region 1.

$$\boldsymbol{B_2 = B_{\infty\ straight\ wire} = \frac{\mu_0 I}{2\pi r}}$$

6f) In region 3 *all* current from region 1 but only *some* current from region 3 is enclosed. Furthermore, because currents run in opposite directions:



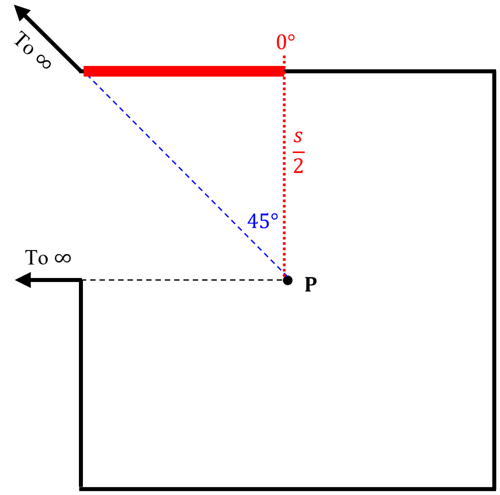
$$B_{tot3} = B_{inner} - B_{outer}$$

$$B_{tot3} = \frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I}{2\pi r} \cdot \frac{A_{amperian}}{A_{region\ 3}}$$

$$\boldsymbol{B_{tot3} = \frac{\mu_0 I}{2\pi r}\left(1 - \frac{(r^2 - 9R^2)}{7R^2}\right) = \frac{\mu_0 I}{2\pi r}\left(\frac{16R^2 - r^2}{7R^2}\right) = \frac{\mu_0 I}{2\pi r} \cdot \frac{16 - \frac{r^2}{R^2}}{7}}$$

6g) The net current enclosed is zero. **No magnetic field.**

7a) Use the right-hand rule on standard coordinates. Clockwise current produces field in $-\hat{k}$ direction.



7b) Consider the thick red line segment shown at right.
Total field should be 7 times the contribution from this segment.

$$B = 7\left(\frac{\mu_0 I}{4\pi a}\left[\sin\theta_f - \sin\theta_i\right]\right)$$

Here $a = \frac{s}{2}$ is perpendicular bisector distance.

Note: since we already know the direction from part a, I will assume angles to the LEFT of the perpendicular bisector are positive. This gives me a positive field *magnitude* as expected.

$$B = 7\left(\frac{\mu_0 I}{4\pi \frac{s}{2}}\left[\sin 45° - \sin 0°\right]\right)$$

$$B = \frac{7\mu_0 I}{2\pi s}\left[\frac{\sqrt{2}}{2} - 0\right]$$

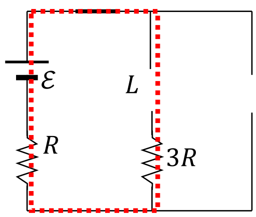
$$B = \frac{7\sqrt{2}\mu_0 I}{4\pi s}$$

$$I = \frac{4\pi B s}{7\sqrt{2}\mu_0}$$

$$\boldsymbol{I \approx 7.47\ \mathrm{A}}$$

8a & 8b) I think these parts are best explain by drawing a set of $t = 0_+$ and $t = \infty$ pictures.
To be clear, these pictures are $t = 0_+$ and $t = \infty$ for the *switch being closed*.
In each circuit the current path is shown with a thick red dotted line.

$t = 0_+$
Just after switch closed

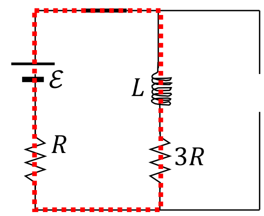


Inductor acts like a *break* just after switch closed.

No current flow.

All voltage across inductor.
$\Delta V_L = \mathcal{E}$

$t = \infty$ (steady state)
Long time after switch closed

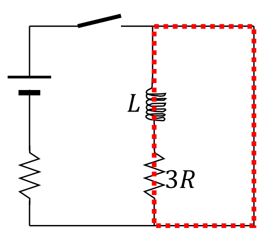


Inductor acts like a *short* in steady state (long after switch closed.

$$i = \frac{\mathcal{E}}{R_{eq}} = \frac{\mathcal{E}}{4R}$$

8c) This was a 1 pointer. Again, probably easiest to start by drawing a set of $t = 0_+$ and $t = \infty$ pictures.
To be clear, these pictures are $t = 0_+$ and $t = \infty$ for the *switch being re-opened*.
In each circuit the current path is shown with a thick red dotted line.

$t = 0_+$
Just after switch re-opened



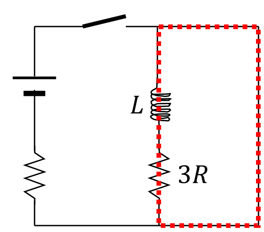
Inductor preserves current just after switch re-opened.

$$i = \frac{\mathcal{E}}{4R}$$

$$\Delta V_{3R} = i(3R) = \frac{3}{4}\mathcal{E}$$

KVL gives $\Delta V_L = \Delta V_{3R}$

$t = \infty$ (steady state)
Long time after switch re-opened



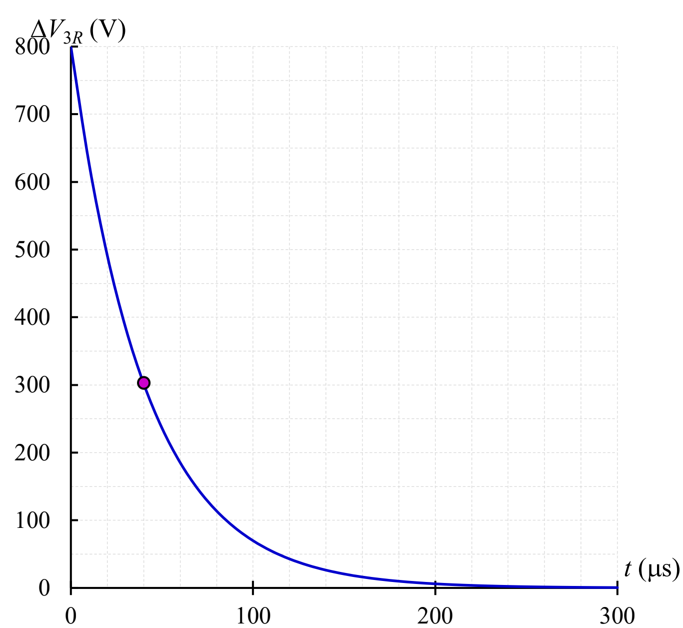
The energy source in this circuit was the magnetic field energy (not a battery).

Eventually all current dies out.

No voltage across anything.

8d) Compare the plot at right to the result of part 8c.

$$\Delta V_{3R\,\mathrm{MAX}} = \frac{3}{4}\mathcal{E} = 800\ \mathrm{V}$$
$$\mathcal{E} = 1067\ \mathrm{V}$$



8e) From part 8c I can write

$$\Delta V_{3R}(t) = \Delta V_{3R\,\mathrm{MAX}} e^{-t/\tau} \quad \text{using } \tau = \frac{L}{3R}$$

From the plot $\Delta V_{3R}(t) = 300$ V at $t = 40$ μs.

$$\frac{\Delta V_{3R}(t)}{\Delta V_{3R\,\mathrm{MAX}}} = e^{-t/\tau}$$

$$-\frac{t}{\left(\frac{L}{3R}\right)} = \ln \frac{\Delta V_{3R}(t)}{\Delta V_{3R\,\mathrm{MAX}}}$$

$$R = -\frac{L}{3t} \ln \frac{\Delta V_{3R}(t)}{\Delta V_{3R\,\mathrm{MAX}}}$$

$$R \approx 10.05\ \Omega$$

Using $\Delta V_{3R}(t) = 500$ V at $t = 20$ μs gave $R \approx 9.7\ \Omega$.

There is no reason to assume the biggest time on a graph just happens to equal 5τ. No credit for such work.