Chapter 24 Solutions

24.1

a) Even though no charge is present at **P** we can still compute the electric *potential*. Don't over think it.

$$V_{total} = V_1 + V_2 + V_3$$
$$V_{total} = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} + \frac{kq_3}{r_3}$$

- Note: it seems easiest to let $q_1 = q$, $q_2 = -2q$, and $q_3 = 3q$.
- The q's are *charges* (not charge magnitudes) and can be positive or negative.
- Electric potential could end up as a positive or negative number even though it is a scalar (just like the scalar quantity temperature can be positive or negative).
- The r's are *distances* and should all be positive.
- For this part of the problem, the r's represent <u>the distances from each charge to the point **P**</u>.
  - $r_1 = \sqrt{(8d)^2 + (12d)^2} \approx 14.42d$
  - $r_2 = 18d$
  - $r_3 = 13d$

$$V_{total} = \frac{k(+q)}{14.42d} + \frac{k(-2q)}{18d} + \frac{k(+3q)}{13d}$$
$$V_{total} = \frac{kq}{d}\left(\frac{1}{14.42} + \frac{-2}{18} + \frac{3}{13}\right)$$
$$\boldsymbol{V_{total} \approx 0.189\frac{kq}{d}}$$

b) To get potential energy we want to use

$$U_{total} = sum\ of\ U's for\ each\ pair\ of\ masses$$
$$U_{total} = U_{12} + U_{13} + U_{23}$$
$$U_{total} = \left(\frac{kq_1q_2}{distance_{12}}\right) + \left(\frac{kq_1q_3}{distance_{13}}\right) + \left(\frac{kq_2q_3}{distance_{23}}\right)$$
$$U_{total} = \left(\frac{kq_1q_2}{r_{12}}\right) + \left(\frac{kq_1q_3}{r_{13}}\right) + \left(\frac{kq_2q_3}{r_{23}}\right)$$

Again, let $q_1 = q$, $q_2 = -2q$, and $q_3 = 3q$.

For this part of the problem, the $r's$ represent distances between each pair of charges.

The distances <u>between the charges</u> are $r_{12} = 10d$, $r_{13} = 13d$, & $r_{23} = 22.20d$.

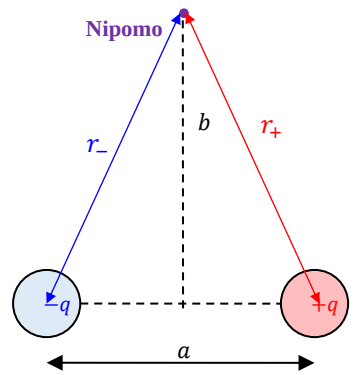
$$U_{total} = \left(\frac{k(q)(-2q)}{10d}\right) + \left(\frac{k(q)(3q)}{13d}\right) + \left(\frac{k(-2q)(3q)}{22.20d}\right)$$

Notice you can factor out $-\frac{kq^2}{d}$ giving

$$U_{total} = \frac{kq^2}{d}\left[\frac{-2}{10} + \frac{3}{13} + \frac{-6}{22.20}\right] = -0.2395\frac{kq^2}{d}$$

24.2

a) Isn't it so nice to not deal with vectors…



$$V_{tot} = V_- + V_+$$

$$V_{tot} = \frac{k(-q)}{r_-} + \frac{k(+q)}{r_+}$$

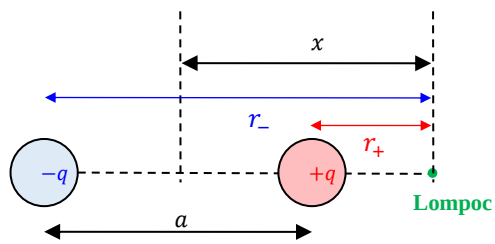
$$V_{tot} = -\frac{kq}{\sqrt{\left(\frac{a}{2}\right)^2 + b^2}} + \frac{kq}{\sqrt{\left(\frac{a}{2}\right)^2 + b^2}}$$

$$V_{tot} = ZERO$$

Electric *potential* (**<u>not</u>** potential *energy*) will cancel any time you have a plus and minus charge equidistant from the point of interest.

Note: on test day you would not need to show any work for part a…you could simply write down $V_{tot} = 0$.

b) Now we are looking at



$$V_{tot} = V_- + V_+$$

$$V_{tot} = \frac{k(-q)}{r_-} + \frac{k(+q)}{r_+}$$

$$V_{tot} = -\frac{kq}{x + \frac{a}{2}} + \frac{kq}{x - \frac{a}{2}}$$

$$V_{tot} = kq\left[\frac{1}{x - \frac{a}{2}} - \frac{1}{x + \frac{a}{2}}\right]$$

Any time you see something like this (similar denominators with subtraction) it is reasonable to try a common denominator. Many times that will clean up some shazizznit…sometimes it won't.

$$V_{tot} = kq\left[\frac{\left(x + \frac{a}{2}\right) - \left(x - \frac{a}{2}\right)}{\left(x - \frac{a}{2}\right)\left(x + \frac{a}{2}\right)}\right]$$

$$V_{tot} = kq\left[\frac{a}{x^2 - \left(\frac{a}{2}\right)^2}\right]$$

c) The first thing I thought to do was to simply plug in $x = 0$ into the above equation.
This gives a non-zero answer for the electric potential!!! This initially freaked me out.

Think about why this should make you uncomfortable:
When $x = 0$ we are located halfway between the charges.
Since a plus and minus charge are equidistant, we know without computation the electric potential is zero.
Another way to see this is to consider the result of part a.
Notice the previous result did not depend on the value of b.
This implies the electric potential is zero anywhere along the y-axis of the problem.
Note: alternatively one could say "the y-axis lies on the 0 V equipotential".

WHAT GIVES?

See the next page to find out what's up…

The issue lies in the way I chose to write up r_+ and r_-. Recall that r_+ and r_- are *distances* and must be *positive*. When $x < a/2$ we would need *absolute values* on the above r_+ and r_- equations.

$$V_{tot} = kq\left[\frac{1}{\left|x-\frac{a}{2}\right|} - \frac{1}{\left|x+\frac{a}{2}\right|}\right] = \begin{cases} kq\left[\dfrac{1}{x-\frac{a}{2}} - \dfrac{1}{x+\frac{a}{2}}\right] = kq\left[\dfrac{a}{x^2-\left(\frac{a}{2}\right)^2}\right] & \boldsymbol{for\ x > \frac{a}{2}} \\ kq\left[\dfrac{1}{\frac{a}{2}-x} - \dfrac{1}{\frac{a}{2}+x}\right] = kq\left[\dfrac{2x}{\left(\frac{a}{2}\right)^2-x^2}\right] & \boldsymbol{for\ 0 < x < \frac{a}{2}} \end{cases}$$

You might now be wondering about the left side…ask me if you care and I'll work it out. It is one of those problems where you have to choose if you want x to be *position* (taking on *negative* values on the left side) or *distance* (still taking on *positive* values on the left side). To be honest, I haven't yet thought it through if the above answers work if you simply let x be a negative number.

d) I don't think you need it for this simple case if you used the common denominator. If $x \gg a$ we can simply ignore the $\frac{a}{2}$ in the basement of the common denominator version to find

$$V_{tot} \approx \frac{kqa}{x^2} \quad \boldsymbol{for\ x \gg \frac{a}{2}}$$

If you didn't see this (or want binomial expansion practice) we could do the following

$$V_{tot} = kq\left[\frac{1}{\left(x-\frac{a}{2}\right)} - \frac{1}{\left(x+\frac{a}{2}\right)}\right]$$

Need a small thing inside each *binomial* (each parenthetical term) to use binomial expansion.
Since we were asked about $x \gg a$ use $\delta = \frac{a}{x} \gg 1$.
To get this, factor out an x from each binomial.

$$V_{tot} = kq\left[\frac{1}{x\left(1-\frac{a}{2x}\right)} - \frac{1}{x\left(1+\frac{a}{2x}\right)}\right]$$

$$V_{tot} = kq\left[\frac{1}{x\left(1-\frac{\delta}{2}\right)} - \frac{1}{x\left(1+\frac{\delta}{2}\right)}\right]$$

$$V_{tot} = \frac{kq}{x}\left[\left(1-\frac{\delta}{2}\right)^{-1} - \left(1+\frac{\delta}{2}\right)^{-1}\right]$$

Now use the binomial expansion to approximate V_{tot}. Use $(1 \pm \delta)^n \approx 1 \pm n\delta + \cdots$

$$V_{tot} \approx \frac{kq}{x}\left[\left(1+(-1)\left(-\frac{\delta}{2}\right)+\cdots\right) - \left(1+(-1)\left(+\frac{\delta}{2}\right)+\cdots\right)\right]$$

$$V_{tot} \approx \frac{kq}{x}\left[\left(1+\frac{\delta}{2}+\cdots\right) - \left(1-\frac{\delta}{2}+\cdots\right)\right]$$

Watch those highlighted signs when you distribute!!!

$$V_{tot} \approx \frac{kq}{x}\left[1+\frac{\delta}{2}+\cdots - 1 + \frac{\delta}{2}+\cdots\right]$$

$$V_{tot} \approx \frac{kq\delta}{x} = \frac{kq\left(\frac{a}{x}\right)}{x}$$

$$V_{tot} \approx \frac{kqa}{x^2} \quad \boldsymbol{for\ x \gg \frac{a}{2}}$$

More on next page…

e) Use

$$U_{total} = sum\ of\ U's for\ each\ pair\ of\ masses$$
$$U_{total} = U_{12}$$
$$U_{total} = \left(\frac{kq_1q_2}{distance_{12}}\right)$$
$$U_{total} = \left(\frac{k(q)(-q)}{a}\right)$$
$$\boldsymbol{U_{total} = -\frac{kq^2}{a}}$$

f) If the potential energy is negative one must hold back the charges while assembling them to keep them from smashing together.

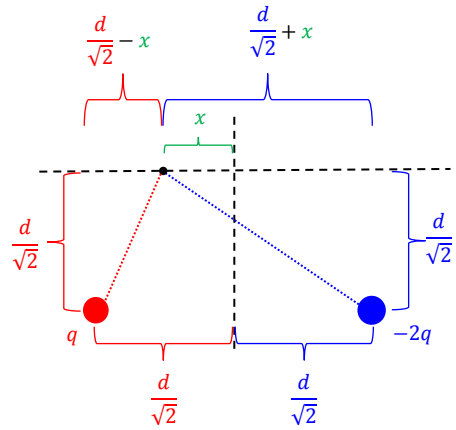
<u>Another way to think about it:</u> the Coulomb force between the two is attractive. While assembling these charges the electric force the positive charge exerts on the negative would point in the same direction the negative charge moves. We could see the work done by the plus charge on the minus charge during assembly is positive (because force points in direction of motion during assembly of charges). Similar logic applies to the force the minus charge exerts on the positive charge.

<u>Another way to say it:</u> The electric field does positive *work* on the system as the charges are assembled. The change in potential energy is opposite the sign of the work done ($W = -\Delta U$). When the charges are far apart there is essentially no energy stored between them. When the electric forces (electric field) brings the two charges together it does positive work. The potential

$$W = -\Delta U$$
$$W = -(U_f - U_i)$$
$$For\ this\ case\ U_i = 0$$
$$W = -(U_f - 0)$$
$$W = -U_f$$
$$U_f = -W$$
$$U_f = -(positive\ number)$$
$$U_f < 0$$

<u>Another way to think about it:</u> A *zombie* must pull outwards on each charge to keep them from smashing into each other. The *zombie* assembling the charges must pull to the right on the plus charge and to the left on the minus charge to prevent them from smashing together. The *zombie* is doing negative work as the forces exerted by the zombie are opposite the direction of motion during assembly. Since the zombie is working *against* the charges during assembly, the electric field must be working for the system. As discussed previously, this implies the change in the electric potential energy (which is opposite in sign to the work done by electric forces/fields) must be negative.

24.3 We expect the point must be a little closer to the positive charge since it is negative.



I chose to define a *distance* ($x > 0$) to the left of the origin.
Re-emphasizing: x is <u>not</u> a position vector; it is a distance.
We expect our final result for x should be positive based on the sizes of the charges.

It is convenient to use $\cos 45° = \sin 45° = \frac{1}{\sqrt{2}}$.

This means the lengths of the sides of the triangles are $\frac{d}{\sqrt{2}}$.

At the end of the problem we can move square roots around or punch crap into a calculator to make it prettier.

We are asked to find the spot where $V_{total} = 0$.

$$0 = V_{total} = V_+ + V_-$$
$$V_+ = -V_-$$
$$\frac{kq}{r_+} = -\frac{k(-2q)}{r_-}$$
$$\frac{1}{r_+} = \frac{2}{r_-}$$
$$\frac{1}{\sqrt{\left(\frac{d}{\sqrt{2}}\right)^2 + \left(\frac{d}{\sqrt{2}} - x\right)^2}} = \frac{\mathbf{2}}{\sqrt{\left(\frac{d}{\sqrt{2}}\right)^2 + \left(\frac{d}{\sqrt{2}} + x\right)^2}}$$

Cross-multiply and square both sides. Don't forget to square that 2!!!

$$\left(\frac{d}{\sqrt{2}}\right)^2 + \left(\frac{d}{\sqrt{2}} + x\right)^2 = \mathbf{4}\left\{\left(\frac{d}{\sqrt{2}}\right)^2 + \left(\frac{d}{\sqrt{2}} - x\right)^2\right\}$$

At this point I can tell a solution (possibly two) will be found. You will get a quadratic for x which gives two solutions. Because we defined x as a distance, any positive solutions are valid.

Let's move on to focus on physics instead of the quadratic formula.

24.4

a) The questions asks about electric *force*. I see parallel plates. From experience I know the electric *field* inside the plates is uniform (if we can ignore fringing fields at the edges of the plates). It would make sense to use

$$F = |q|E = eE$$

It is worth emphasizing e is the *magnitude* of charge on the proton 1.602×10^{-19} C. Also notice this result is the *magnitude* of the force. Think about the direction using *plus* charges *go with* the electric field while *minus* charges *go against* the electric field. Force vector is to the right.

$$\vec{F} = eE\hat{\imath} \approx (1.6 \times 10^{-16}\ \text{N})\hat{\imath}$$

Side notes on how you would get E if not given to you. To get E I know, from Gauss's law

Type of plate	$\boldsymbol{\sigma}$ **on single surface**	$\boldsymbol{E}$ **in terms of** $\boldsymbol{\sigma}$	$\boldsymbol{E}$ **in terms of** $\boldsymbol{Q}$
Insulator with Q on a *single* surface	$\sigma = \frac{Q_{total}}{A_{total}}$	$E = \frac{\sigma}{2\epsilon_0}$	$E = \frac{Q}{2A_{total}\epsilon_0}$
Conductor with Q spread out over *both* surfaces	$\sigma_{conductor} = \frac{Q_{total}}{2A_{total}}$	$E = \frac{\sigma_{conductor}}{\epsilon_0}$	$E = \frac{Q}{2A_{total}\epsilon_0}$

Notice the electric field magnitudes seem slightly different in the first two columns but end up the same in the third. Most textbooks will write

$$E = \frac{\sigma}{2\epsilon_0} \quad where\ \sigma = \frac{Q_{total}}{A_{total}}$$

Sometimes parallel plate electric field magnitude is related to potential difference (voltage) instead. That formula is

$$E = \frac{|\Delta V|}{d} \quad where\ d\ is\ plate\ spacing$$

b) First I'll do it the hard way to set you up for later problems…then tell you the shortcut. From the definition of work in physics we know

$$W = \int_i^f \vec{F} \cdot d\vec{s}$$

For a particle moving to the right we know $d\vec{s} = dx\hat{\imath}$.

$$W = \int_i^f (eE\hat{\imath}) \cdot (dx\hat{\imath})$$

$$W = \int_i^f eE(\hat{\imath} \cdot \hat{\imath})\, dx$$

$$W = eE \int_i^f dx$$

$$W = eE(\Delta x)$$

$$W = eEd \approx 8 \times 10^{-18}\ \text{J}$$

Shortcut: maybe you recall when the force is constant we can usually pull things out and use

$$W = \vec{F} \cdot (displacement) = \vec{F} \cdot \Delta\vec{r} = Fd\cos\theta$$

c) Use the work energy theorem

$$W = \Delta K$$

$$W = K_f - K_i$$

Particle initially at rest. This means $K_i = \frac{1}{2}mv_i^2$ is zero!

$$W = K_f \approx 8 \times 10^{-18}\ \text{J}$$

d) Now use

$$W = \frac{1}{2}mv_f^2$$

$$v_f = \sqrt{\frac{2W}{m}} = \sqrt{\frac{2eEd}{m}} \approx 9.8 \times 10^4 \frac{\text{m}}{\text{s}} = 98 \frac{\text{km}}{\text{s}}$$

Important note: since $W = -\Delta U$ we often see this written in textbooks as

$$v_f = \sqrt{-\frac{2\Delta U}{m}}$$

Worth mentioning: The speed of light is $3 \times 10^8 \frac{\text{m}}{\text{s}}$. We can typically ignore corrections associated with relativity when out speeds are less than 1% of the speed of light.

e) The *potential difference* between the plates is ΔV. This is **<u>not</u>** the same thing as the *difference in potential energy* ΔU. The two are related by

$$\Delta U = q\Delta V$$

$$\Delta V = \frac{\Delta U}{q}$$

Recall $W = -\Delta U$ (that minus sign is pretty important).

$$\Delta V = \frac{-W}{q}$$

$$\Delta V = \frac{-(eEd)}{e} = -Ed = -50 \text{ Volts} = -50 \text{ V}$$

Comment: watch out for the subtle difference between the variable V (italicized) and the units V (not italicized).

f) If negative plate is at ground (0 V) we know the positive plate is at $+50$ V.

g) The goal of this part of the problem was to stress the subtle difference between

$$\Delta U = q\Delta V \quad \text{potential energy difference BETWEEN TWO POINTS}$$

$$U = qV \quad \text{potential energy AT A SINGLE POINT}$$

$$U_i = qV_i = eEd =$$

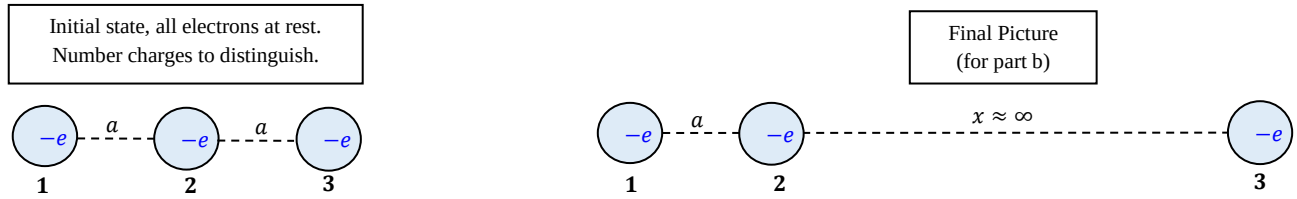
Note: another style worth knowing is to use

$$\Delta U = -\Delta K$$

h) Before mentioned what changes, perhaps it is worth noting was *doesn't* change. Switching to an electron has no effect on the electric *field* ($\vec{E}$) or the electric *potential* (V) between the plates. The electric force *magnitude* (F) will not change but the direction of the electric force ($\vec{F}$) does change! The sign electric potential *difference* (sign of ΔV) changes because we are now going from the low plate. Similarly, the electric potential energy difference (sign of ΔU) changes. Lastly, because electrons are much less massive, the final impact speed should be much larger (about 43 times larger)!

Note: an excellent approximation is $m_p \approx 2000m_e$

24.5



$$U_{total} = sum\ of\ U'sfor\ each\ pair\ of\ masses$$

a) First consider the *initial* state.

$$U_i = U_{12i} + U_{13i} + U_{23i}$$

$$U_i = \left(\frac{kq_1q_2}{distance_{12i}}\right) + \left(\frac{kq_1q_3}{distance_{13i}}\right) + \left(\frac{kq_2q_3}{distance_{23i}}\right)$$

$$U_i = \left(\frac{kq_1q_2}{r_{12i}}\right) + \left(\frac{kq_1q_3}{r_{13i}}\right) + \left(\frac{kq_2q_3}{r_{23i}}\right)$$

Note: $q_1 = q_2 = q_3 = -e$. Remember e is *magnitude* of electron charge…

The distances between the masses are $r_{12i} = a$, $r_{13i} = 2a$, & $r_{23i} = a$. Therefore

$$U_i = \left(\frac{k(-e)(-e)}{a}\right) + \left(\frac{k(-e)(-e)}{2a}\right) + \left(\frac{k(-e)(-e)}{a}\right)$$

$$U_i = \frac{ke^2}{a}\left(1 + \frac{1}{2} + 1\right)$$

$$\boldsymbol{U_i = \frac{5ke^2}{2a}}$$

b) The main idea here is to relate kinetic and potential energy. Because electrical forces are conservative we know we can write the work done by the electric force as potential energy and use

$$\Delta K = -\Delta U$$

Personally, I like to rewrite this by splitting up the initial and final states like this:

$$U_i + K_i = U_f + K_f$$

Now consider the *final* state.

$$U_f = U_{12f} + U_{13f} + U_{23f}$$

$$U_f = \left(\frac{kq_1q_2}{r_{12f}}\right) + \left(\frac{kq_1q_3}{r_{13f}}\right) + \left(\frac{kq_2q_3}{r_{23f}}\right)$$

This time the distances between the masses are $r_{12i} = a$, $r_{13i} = x + a \approx \infty$, & $r_{23i} = x \approx \infty$. Therefore

$$U_f = \left(\frac{ke^2}{a}\right) + \left(\frac{ke^2}{x+a}\right) + \left(\frac{ke^2}{x}\right)$$

The last two terms on the right hand side are essentially zero!

$$\boldsymbol{U_f = \frac{ke^2}{a}}$$

Now use

$$U_i + K_i = U_f + K_f$$

$$\boldsymbol{\frac{5ke^2}{2a}} + 0 = \boldsymbol{\frac{ke^2}{a}} + \frac{1}{2}m_e v_f^2$$

More on next page…

From the previous page…

$$\frac{\mathbf{5ke^2}}{\mathbf{2a}} + 0 = \frac{\mathbf{ke^2}}{\mathbf{a}} + \frac{1}{2}m_e v_f^2$$

From there you can solve for v_f. You should find

$$v_f = \sqrt{\frac{3ke^2}{m_e a}}$$

Another really nice style is to do things in this order

$$\Delta K = -\Delta U$$

$$K_f - K_i = -\Delta U$$

$$\frac{1}{2}m_e v_f^2 - 0 = -\Delta U$$

$$v_f = \sqrt{-\frac{2\Delta U}{m_e}}$$

$$v_f = \sqrt{-\frac{2(U_f - U_i)}{m_e}}$$

$$v_f = \sqrt{-\frac{2\left(\left(\frac{\mathbf{ke^2}}{\mathbf{a}}\right) - \left(\frac{\mathbf{5ke^2}}{\mathbf{2a}}\right)\right)}{m_e}}$$

$$v_f = \sqrt{-2\frac{ke^2}{m_e a}\left(\mathbf{1} - \frac{\mathbf{5}}{\mathbf{2}}\right)}$$

$$v_f = \sqrt{-2\frac{ke^2}{m_e a}\left(-\frac{\mathbf{3}}{\mathbf{2}}\right)}$$

$$v_f = \sqrt{\frac{3ke^2}{m_e a}}$$

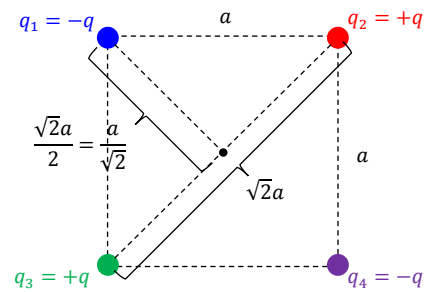
Notice how crucial it is to keep careful track of signs!!!

BONUS Q: Why can't we use kinematics equations like $\Delta x = \frac{1}{2}at^2 + v_i t$ for this problem? Why is it ok to apply this equation for a point charge moving through parallel plates but not for point charges moving apart form each other?

As point charges move away from each other, the forces they exert on each other will change. Remember: for point charges the force is proportional to $\frac{1}{r^2}$!!! Parallel plates, on the other hand, have uniform electric fields in between the plates. If the electric field is constant, so too is the force on and thus the acceleration of a point charge moving between the plates.

24.6

a) The charges are equidistant from the center. As far as electric *potential* is concerned, we don't have to worry about any vector crap. We have an equal amount of positive and negative charges. **The electric *potential* is zero at the center.**



It is worth running through the following to get used to my style.

$$V_{tot} = V_1 + V_2 + V_3 + V_4$$

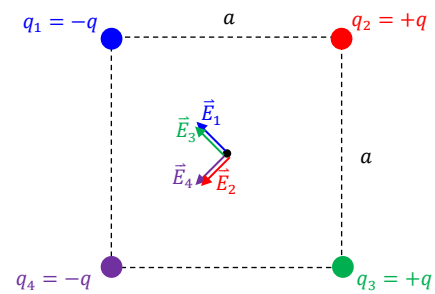
$$V_{tot} = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} + \frac{kq_3}{r_3} + \frac{kq_4}{r_4}$$

$$V_{tot} = -\frac{kq}{(a/\sqrt{2})} + \frac{kq}{(a/\sqrt{2})} - \frac{kq}{(a/\sqrt{2})} + \frac{kq}{(a/\sqrt{2})}$$

$$V_{tot} = 0$$

b) By symmetry the electric *field* is also zero at the center.

c) If you switch the bottom two charges the picture looks like the lower figure at right. Notice the electric potential is unchanged but the electric fields no longer cancel! By symmetry the *vertical* components of $\vec{E}$ cancel but not the *horizontal* components. In fact, we could find the horizontal component of any one of the fields and multiply by 4 to get the total electric field since all four cause the same field at the center. In equation form:



$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4$$

$$\vec{E}_{tot} = 4\vec{E}_{1x}$$

Why do all the $\hat{r}$ crap for this one; use *magnitudes* think about direction from picture with SOH CAH TOA.

$$\vec{E}_{tot} = 4\vec{E}_1 \cos 45°$$

$$\vec{E}_{tot} = 4\left(E_1(-\hat{\imath})\right) \cos 45°$$

$$\vec{E}_{tot} = 4\left(\left|\frac{k(-q)}{\left(\frac{a}{\sqrt{2}}\right)^2}\right|(-\hat{\imath})\right)\frac{\sqrt{2}}{2}$$

$$\boldsymbol{\vec{E}_{tot} = \frac{4\sqrt{2}kq}{a^2}(-\hat{\imath})}$$

d) If you had switched the left two charges (instead of the bottom two) the reasoning is similar. The electric potential is still zero at the center, the magnitude of the field is identical to part c, but the direction would be downwards instead of to the left ($-\hat{\jmath}$ instead of $-\hat{\imath}$).

24.7 Draw a before and after picture. Remember, we are assuming we assume the green positive charge is held stationary.

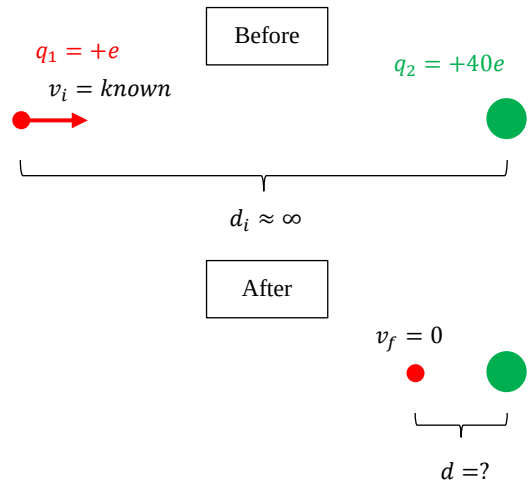
$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}m_p v_i^2 + \frac{kq_1q_2}{r_{12i}} = \frac{1}{2}m_p v_f^2 + \frac{kq_1q_2}{r_{12f}}$$

$$\frac{1}{2}m_p v_i^2 + \frac{kq_1q_2}{\infty} = 0 + \frac{kq_1q_2}{d}$$

$$\frac{1}{2}m_p v_i^2 = \frac{kq_1q_2}{d}$$

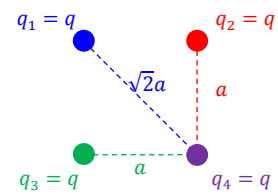
$$d = \frac{2kq_1q_2}{m_p v_i^2}$$



24.8

a) See figures at right. Since each charge is identical, we assume they have the same mass m. Note: by symmetry we expect each charge to be pushed diagonally outwards from the center of the square. This means we will have 4 identical kinetic energy terms of $\frac{1}{2}mv_f^2$.

Before Release



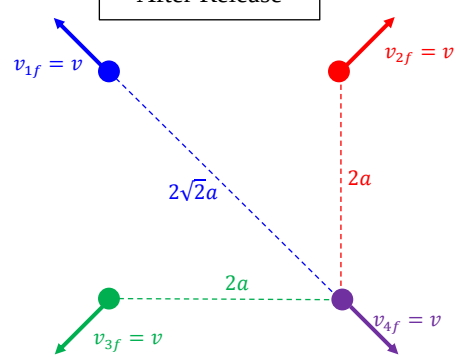
$$\Delta K = -\Delta U$$
$$K_f - K_i = -\Delta U$$
$$\mathbf{4}\left(\frac{1}{2}mv_f^2\right) - 0 = -\Delta U$$
$$\boldsymbol{v_f} = \sqrt{-\frac{\boldsymbol{\Delta U}}{\mathbf{2}\boldsymbol{m}}}$$

To figure out the initial potential energy one must consider each **pair** of charges. My way to not miss any pairs is to first do every possible paring with 1, then every pairing with two (but don't double count 1 & 2), then every pairing with 3 (but don't double count 1 & 3 and 2 & 3), etc.

$$U_i = U_{12i} + U_{13i} + U_{14i} + U_{23i} + U_{24i} + U_{34i}$$

Note: if you care, the number of pairs is given by "N choose 2"…ask your probability instructor sometime. In this case 4 choose 2 $= \binom{4}{2} = \frac{4!}{2!(4-2)!} = 6$.

After Release



From the picture I hope you can see the important distances are

$$r_{14i} = \sqrt{2}a \qquad r_{24i} = a \qquad r_{34i} = a$$

At this point the color coding breaks down but the other important distances are

$$r_{12i} = a \qquad r_{13i} = a \qquad r_{23i} = \sqrt{2}a$$

Finally, we have

$$U_i = U_{12i} + U_{13i} + U_{14i} + U_{23i} + U_{24i} + U_{34i}$$
$$U_i = \frac{kq^2}{r_{12i}} + \frac{kq^2}{r_{13i}} + \frac{kq^2}{r_{14i}} + \frac{kq^2}{r_{23i}} + \frac{kq^2}{r_{24i}} + \frac{kq^2}{r_{34i}}$$
$$U_i = kq^2\left(\frac{1}{r_{12i}} + \frac{1}{r_{13i}} + \frac{1}{r_{14i}} + \frac{1}{r_{23i}} + \frac{1}{r_{24i}} + \frac{1}{r_{34i}}\right)$$
$$U_i = kq^2\left(\frac{1}{a} + \frac{1}{a} + \frac{1}{\sqrt{2}a} + \frac{1}{\sqrt{2}a} + \frac{1}{a} + \frac{1}{a}\right)$$
$$U_i = \frac{kq^2}{a}\left(4 + \frac{2}{\sqrt{2}}\right)$$
$$U_i \approx 5.414\frac{kq^2}{a}$$

To calculate U_f you could grind it out directly in a similar manner. Or, thinking a bit, hopefully you can see if each distance doubles AND the denominator is proportional to distance (not distance *squared*) we can say $U_f = \frac{1}{2}U_i$ in this special case! Therefore

$$U_f = \frac{U_i}{2} = \frac{kq^2}{a}\left(2 + \frac{1}{\sqrt{2}}\right) \approx 2.707\frac{kq^2}{a}$$

Continues next page…

Notice, in this special case where $U_f = \frac{1}{2}U_i$, we find

$$\Delta U = U_f - U_i$$
$$\Delta U = \frac{1}{2}U_i - U_i$$
$$\Delta U = -\frac{1}{2}U_i = -U_f$$

Read that last line closely to make sure you're good with it.
Plugging into our bold equation from the top gives

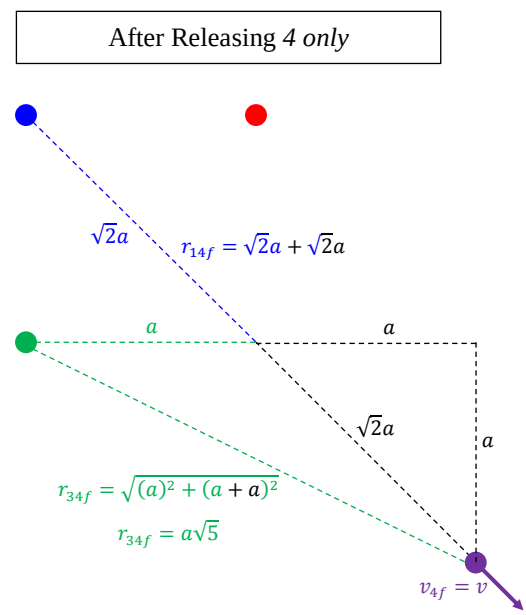
$$\boldsymbol{v_f = \sqrt{-\frac{\Delta U}{2m}}}$$
$$v_f = \sqrt{-\frac{(-U_f)}{2m}}$$

Signs cancel; looks like we are on the right track.

$$v_f = \sqrt{\frac{U_f}{2m}}$$
$$\boldsymbol{v_f = \sqrt{\frac{kq^2}{2ma}\left(2 + \frac{1}{\sqrt{2}}\right)} \approx 1.163\sqrt{\frac{kq^2}{ma}}}$$

b) Each charge will experience a repulsive force if all charges are positive. Likewise, each charge will feel a repulsive force if all charges are negative. As long as all charges are the same they will all repel outwards along the diagonals of the triangle.

c) Sometimes students erroneously calculate the force using Coulomb's law to get acceleration. This value of acceleration is correct for the *initial* acceleration of the system. After the charges all move the charge spacings are different. Thus the forces on, and accelerations of, each charge also change. The equation $v_f^2 = v_i^2 + 2a\Delta x$ is only valid for constant acceleration!!!



d) It is awfully tempting to think one could simply divide the kinetic energy by 4 in line one and call it a day. It is also incorrect…

$$\Delta K = -\Delta U$$
$$K_f - K_i = -\Delta U$$
$$\left(\frac{1}{2}mv_f^2\right) - 0 = -\Delta U$$
$$\boldsymbol{v_{\substack{f \\ only\ 4}} = \sqrt{-\frac{2\Delta U}{m}}}$$
$$U_i = U_{12i} + U_{13i} + U_{14i} + U_{23i} + U_{24i} + U_{34i}$$

In this case there is no change in the locations of 1, 2, and 3. That means

$U_{12i} = U_{12f}$ $\quad$ $U_{13i} = U_{13f}$ $\quad$ $U_{23i} = U_{23f}$

Since we are only concerned with ΔU all terms without a 4 in the subscript to drop out!

$$\Delta U = (U_{14f} - U_{14i}) + (U_{24f} - U_{24i}) + (U_{34f} - U_{34i})$$

<u>Initial</u> distances between charges are: $r_{14i} = \sqrt{2}a$ $\quad$ $r_{24i} = a$ $\quad$ $r_{34i} = a$

<u>Final</u> distances between charges are: $r_{14f} = 2\sqrt{2}a$ $\quad$ $r_{24f} = \sqrt{5}a$ $\quad$ $r_{34f} = \sqrt{5a}$

More on next page…

$$\Delta U = kq^2\left\{\left(\frac{1}{r_{14f}} - \frac{1}{r_{14i}}\right) + \left(\frac{1}{r_{24f}} - \frac{1}{r_{24i}}\right) + \left(\frac{1}{r_{34f}} - \frac{1}{r_{34i}}\right)\right\}$$

$$\Delta U = \frac{kq^2}{a}\left\{\left(\frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{5}} - 1\right) + \left(\frac{1}{\sqrt{5}} - 1\right)\right\}$$

This is getting ugly. Might as well compute the decimal now to find something more meaningful…

$$\Delta U = -1.459\frac{kq^2}{a}$$

$$\boldsymbol{v_{\substack{f \\ only4}} = \sqrt{-\frac{2\Delta U}{m}}}$$

$$v_{\substack{f \\ only4}} \approx 1.708\sqrt{\frac{kq^2}{ma}}$$

Whew! All that work to answer one thing I was curious about. I wanted to see the factor by which the speed increases if one charge (versus all 4) were released from rest. One finds

$$\frac{v_{\substack{f \\ only4}}}{v_{\substack{f \\ all4}}} \approx \frac{1.708\sqrt{\frac{kq^2}{ma}}}{1.163\sqrt{\frac{kq^2}{ma}}} \approx 1.47$$

Notice charge 4 moves about 47% faster after traveling the same distance compared to all charges in the square moving. Not what I expected at all!

If you really want a challenge, redo all of this for and arbitrary distance x.
Compare the velocity 4 when only it moves distance x to the velocity 4 when all four move distance x.
Think: do you think the x will drop out or not? Is the ratio monotonically increasing, monotonically decreasing, or something else with perhaps a max or min?

I suppose you could imagine two similar set-ups released simultaneously and think of this as a race to infinity. In this scenario the q_4 that was released by itself would be racing to infinity at a greater rate than the q_4 released with all four simultaneously. If x doesn't drop out in the ratio it means the ratio of the speeds is not constant. Is this meaningful? I don't know, but for some reason I found it fun to think about.

BTW – Can you image using energy conservation in a code (instead of force and momentum)?
I'm not sure how it would go down, but probably something a bit like this:

i. Update the potential energy
ii. Use the update potential to update kinetic energy (and speed).
iii. Use this new speed to update position.
iv. Increment time and repeat the loop.

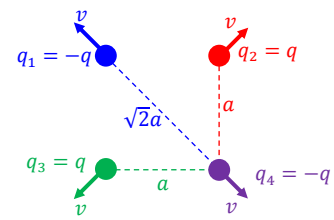
Notice: this would only work when you know the direction of travel due to symmetry. There are more advanced techniques which might make this method doable, even when you *don't* know the direction by symmetry, but those are beyond the scope of this course.

24.9

a) Your problem, your guess.

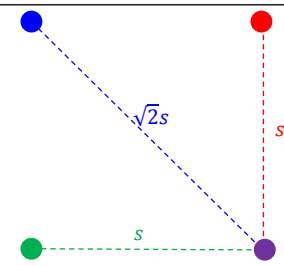
b) See figures at right. Since each charge is identical, we assume they have the same mass m. Note: by symmetry we expect each charge to be pushed diagonally *inwards* towards the center of the square. This can be shown by doing Coulomb's law on each charge.

Before Release



We expect the shape will remain a square that gets larger upon being released. Since the force on each charge is inwards for any arbitrary square, the charges should move away from each other while slowing down. Assuming the initial speed is slow enough, the charges will eventually stop, reverse direction, gradually speed up and eventually smash into each other with nearly zero separation.

After release net force towards center slows and stops each charge



We will still have 4 identical kinetic energy terms, $\frac{1}{2}mv^2$, but this time that is the *initial* kinetic energy.

Again, for the initial states we have

$$r_{14i} = \sqrt{2}a \qquad r_{24i} = a \qquad r_{34i} = a$$
$$r_{12i} = a \qquad r_{13i} = a \qquad r_{23i} = \sqrt{2}a$$

Just before impact charges have nearly zero separation and very high speed

Doing the same process as the previous square we once again find

$$U_i = U_{12i} + U_{13i} + U_{14i} + U_{23i} + U_{24i} + U_{34i}$$
$$U_i = \frac{kq_1q_2}{r_{12i}} + \frac{kq_1q_3}{r_{13i}} + \frac{kq_1q_4}{r_{14i}} + \frac{kq_2q_3}{r_{23i}} + \frac{kq_2q_4}{r_{24i}} + \frac{kq_3q_4}{r_{34i}}$$
$$U_i = -\frac{kq^2}{r_{12i}} - \frac{kq^2}{r_{13i}} + \frac{kq^2}{r_{14i}} + \frac{kq^2}{r_{23i}} - \frac{kq^2}{r_{24i}} - \frac{kq^2}{r_{34i}}$$
$$U_i = kq^2\left(-\frac{1}{r_{12i}} - \frac{1}{r_{13i}} + \frac{1}{r_{14i}} + \frac{1}{r_{23i}} - \frac{1}{r_{24i}} - \frac{1}{r_{34i}}\right)$$
$$U_i = kq^2\left(-\frac{1}{a} - \frac{1}{a} + \frac{1}{\sqrt{2}a} + \frac{1}{\sqrt{2}a} - \frac{1}{a} - \frac{1}{a}\right)$$
$$U_i = \frac{kq^2}{a}\left(-4 + \frac{2}{\sqrt{2}}\right)$$
$$U_i = \frac{2kq^2}{a}\left(-2 + \frac{1}{\sqrt{2}}\right)$$
$$U_i \approx -2.586\frac{kq^2}{a}$$

The only thing changing when the charges move apart is the size of the square (from a to s). That means

$$U_f \approx -2.586\frac{kq^2}{s}$$

The *change* in potential energy is thus

$$\Delta U = U_f - U_i = -2.586kq^2\left(\frac{1}{s} - \frac{1}{a}\right)$$

Note the following: $s > a$ if the charges are coming together. We see $\Delta U > 0$. If potential energy is increasing it implies kinetic energy is decreasing (the charges are slowing). This matches our reasoning. More next page…

This time our energy relationship becomes

$$\Delta K = -\Delta U$$
$$K_f - K_i = -(U_f - U_i)$$

$$0 - \mathbf{4}\left(\frac{1}{2}mv_i^2\right) = -\left(-2.586kq^2\left(\frac{1}{s} - \frac{1}{a}\right)\right)$$

Cancelling the red minus sign with the black one gives

$$-2mv^2 = 2.586kq^2\left(\frac{1}{s} - \frac{1}{a}\right)$$

$$-\frac{2mv^2}{2.586kq^2} = \frac{1}{s} - \frac{1}{a}$$

$$\frac{1}{s} = \frac{1}{a} - 0.7735\frac{mv^2}{kq^2}$$

$$s = \frac{1}{\frac{1}{a} - 0.7735\frac{mv^2}{kq^2}}$$

Multiply each term in the fraction by a for style

$$s = a\frac{1}{1 - 0.7735\frac{mv^2a}{kq^2}}$$

You might notice that crap in the denominator if I rewrite it as

$$\boldsymbol{s = a\frac{1}{1 + \frac{K_i}{U_i}}}$$

Watch out! The minus sign change is correct in that last line…$U_i = -2.586\frac{kq^2}{a}$

Alternate style:

$$K_i + U_i = K_f + U_f$$

$$K_i + U_i = U_f$$

$$K_i + U_i = -2.586\frac{kq^2}{s}$$

$$\frac{1}{K_i + U_i} = \frac{s}{-2.586kq^2}$$

$$s = \frac{-2.586kq^2}{K_i + U_i}$$

Be clever and multiply the numerator by 1 in the form $\frac{a}{a}$

$$s = \frac{a\left(\frac{-2.586kq^2}{a}\right)}{K_i + U_i} = a\frac{(K_i)}{K_i + U_i} = a\frac{1}{1 + \frac{K_i}{U_i}}$$

GOING FURTHER on the next page:

GOING FURTHER:

Notice the charges *might* initially be moving so fast they fly apart forever. The minimum initial speed required to permanently dissociate these charges we can call the escape speed v_{esc}. At escape speed the charges are just barely able to reach infinitely far apart from each other.

$$\Delta K = -\Delta U$$

$$K_f - K_i = -(U_f - U_i)$$

$$0 - \mathbf{4}\left(\frac{1}{2}mv_i^2\right) = -\left(-2.586kq^2\left(\frac{1}{s} - \frac{1}{a}\right)\right)$$

We determine v_{esc} by setting all the r_f's to ∞ and all the v_i's to v_{esc}. In this case the $\frac{1}{s}$ term drops out…if the charges are able to escape the final square has size $s = \infty$.

$$-\mathbf{4}\left(\frac{1}{2}mv_{esc}^2\right) = -\left(-2.586kq^2\left(-\frac{1}{a}\right)\right)$$

$$2mv_{esc}^2 = \left(-2.586kq^2\left(-\frac{1}{a}\right)\right)$$

$$2mv_{esc}^2 = 2.586\frac{kq^2}{a}$$

$$v_{esc} = 1.137\sqrt{\frac{kq^2}{ma}}$$

For any initial speed less than v_{esc} the charges will come crashing back together. This is analogous to throwing a ball up in the air from the surface of the earth. What goes up comes down (unless $v_i < v_{esc}$)!

c) **What about when they smash into each other?** At this point the center-to-center distance between charges is equal to the diameter of the ions. This distance of separation is so tiny we would interpret them as having nearly negative infinity for the electric potential energy. This implies each ion gains nearly infinite kinetic energy. This wouldn't make sense in real life. The situation violates special relativity. Since we don't know special rel, let's move on to something else.

d) If protons and electrons were used the positive charges would be about 2000 times more massive than the negative charges. The initial forces would not change but the initial accelerations would. The less massive electrons would accelerate towards the center at a larger rate and would slow down sooner. I believe the problem would still always be symmetric and the charges would still all move along the diagonals of the square, but the protons would go farther away from the center than the electrons. At this point perhaps the accelerations get wildly different do the wildly different masses. Pretty complicated. To solve this, I would create a numerical simulation.

24.10

a) We want to use energy methods. The initial speeds are all zero. Bottom two charges stay in place. Top charge travels really far away. Once really far away, the top charge is travelling with speed v_f.

$$\Delta K = -\Delta U$$
$$K_f - K_i = -\Delta U$$
$$\left(\frac{1}{2}mv_f^2\right) - 0 = -\Delta U$$
$$v_f = \sqrt{-\frac{2\Delta U}{m}}$$

First determine the *initial* potential energy:

$$U_i = U_{12i} + U_{13i} + U_{23i}$$
$$U_i = \frac{k(-q)^2}{r_{12i}} + \frac{k(-q)^2}{r_{13i}} + \frac{k(-q)^2}{r_{23i}}$$
$$U_i = kq^2\left(\frac{1}{r_{12i}} + \frac{1}{r_{13i}} + \frac{1}{r_{23i}}\right)$$
$$U_i = kq^2\left(\frac{1}{a} + \frac{1}{a} + \frac{1}{a}\right)$$
$$U_i = 3\frac{kq^2}{a}$$

The *final* potential energy, when all three charges are far away from each other, should be zero:

$$U_f = U_{12f} + U_{13f} + U_{23f}$$
$$U_f = \frac{k(-q)^2}{r_{12f}} + \frac{k(-q)^2}{r_{13f}} + \frac{k(-q)^2}{r_{23f}}$$
$$U_f = kq^2\left(\frac{1}{r_{12f}} + \frac{1}{r_{13f}} + \frac{1}{r_{23f}}\right)$$
$$U_f \approx kq^2\left(\frac{1}{\infty} + \frac{1}{\infty} + \frac{1}{a}\right)$$
$$U_f \approx \frac{kq^2}{a}$$

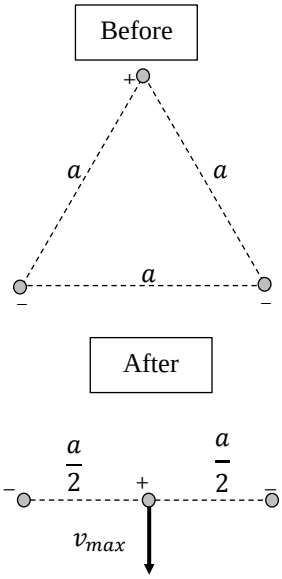
Note: I' m assuming the top charge is charge 1 here.

The change in potential energy is thus

$$\Delta U = U_f - U_i$$
$$\Delta U = \frac{kq^2}{a} - 3\frac{kq^2}{a} = -2\frac{kq^2}{a}$$
$$v_f = \sqrt{-\frac{2\left(-2\frac{kq^2}{a}\right)}{m}}$$
$$v_f = \sqrt{\frac{4kq^2}{ma}}$$

24.11

a) We want to use energy methods. Think, maximum kinetic energy occurs when potential energy is a minimum. Max kinetic will also be max speed. This occurs when the top charge has traveled to the midpoint on the line connecting the bottom two charges.



Think of it another way: the upper charge experiences a net downwards force due to Coulomb's law and will accelerate downwards. Once the positive charge is midway between the two lower negative charges there is no net force on it. Once the positive is below the two negative charges, there will be an upwards force which will slow it down. The max speed must occur when the positive charge is halfway between the two negative charges.

Doing the energy problem

$$\Delta K = -\Delta U$$
$$K_f - K_i = -\Delta U$$
$$\left(\frac{1}{2}mv_{max}^2\right) - 0 = -\Delta U$$
$$v_{max} = \sqrt{-\frac{2\Delta U}{m}}$$

I will assume the top charge is $q_1 = q$ and the lower charges are $q_2 = q_3 = -q$.
First determine the *initial* potential energy:

$$U_i = U_{12i} + U_{13i} + U_{23i}$$
$$U_i = \frac{k(+q)(-q)}{r_{12i}} + \frac{k(+q)(-q)}{r_{13i}} + \frac{k(-q)^2}{r_{23i}}$$
$$U_i = kq^2\left(\frac{-1}{r_{12i}} + \frac{-1}{r_{13i}} + \frac{1}{r_{23i}}\right)$$
$$U_i = kq^2\left(-\frac{1}{a} - \frac{1}{a} + \frac{1}{a}\right)$$
$$U_i = -\frac{kq^2}{a}$$

The *final* potential energy, when the plus charge is exactly halfway between the two negative charges, is

$$U_f = U_{12f} + U_{13f} + U_{23f}$$
$$U_f = \frac{k(+q)(-q)}{r_{12i}} + \frac{k(+q)(-q)}{r_{13i}} + \frac{k(-q)^2}{r_{23i}}$$
$$U_f = kq^2\left(\frac{-1}{r_{12f}} + \frac{-1}{r_{13f}} + \frac{1}{r_{23f}}\right)$$
$$U_f = kq^2\left(\frac{-1}{a/2} + \frac{-1}{a/2} + \frac{1}{a}\right)$$
$$U_f = kq^2\left(\frac{-2}{a} + \frac{-2}{a} + \frac{1}{a}\right)$$
$$U_f = -3\frac{kq^2}{a}$$

Notice: U_f is more negative than U_i. We have decreased potential energy. This seems reasonable as we know kinetic energy has increased.
The *change* in potential energy is thus

$$\Delta U = U_f - U_i$$

$$\Delta U = -3\frac{kq^2}{a} - \left(-\frac{kq^2}{a}\right)$$

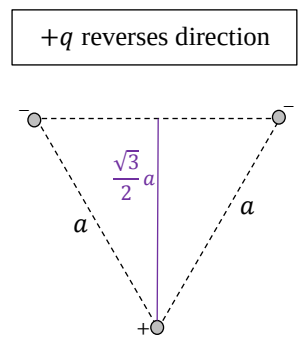
$$\Delta U = -2\frac{kq^2}{a}$$

$$v_{max} = \sqrt{-\frac{2\left(-2\frac{kq^2}{a}\right)}{m}}$$

$$v_{max} = \sqrt{\frac{4kq^2}{ma}}$$

WOW! This is exactly the same as the final speed of the last problem…far out! I wonder if there is a connection or if it is just coincidence…I have no clue but it is cool.

b) Energy is conserved in this scenario as we typically assume no air resistance for ions. We know the positive charge should move an identical distance below the two charges. Basically, it will look like the figure at right. The plus charge can't go any farther below than this. When the charges form the upside down triangle we have the same amount of potential energy as in the initial state. If $U_f = U_i$ there is no ΔU and thus the plus charge isn't moving anymore!



c) The plus charge will oscillate up and down if alignment is perfect.

d) The motion is probably NOT simple harmonic. Simple harmonic oscillators have restoring forces proportional to x (spring force magnitude is $F = kx$). In this case the force magnitude is something ugly like

$$-\frac{2kq^2y\hat{j}}{(y^2 + a^2)^{3/2}}$$

where y is the vertical position relative to the line between the two negative charges.
Note: if the initial height was *tiny* (instead of being on an equilateral triangle) the motion *would be* simple harmonic and obey $y = A\cos\omega t$. I suspect you might be able to write the velocity as a function of y by doing an energy problem for arbitrary y from the midline and do separation of variables to determine the position as a function of time…but maybe this is horrible and it is easier to learn about the motion of the particle with a simulation?

e) If you run a simulation you will see the positive charge drift immediately towards one of the negative charges and appear to orbit. However, if you run the simulation long enough you can observe all kinds of bizarre orbits. I don't know if the motion is chaotic or not but perhaps a really smart person could figure that out. I *suspect* any practical system would be chaotic (minute change in initial alignment can cause wildly different motion) but I do not know this for sure.

24.12 See figures at right.

$$\Delta K = -\Delta U$$
$$K_f - K_i = -\Delta U$$
$$\frac{1}{2}m_e v_f^2 - 0 = -\Delta U$$
$$v_f = \sqrt{-\frac{2\Delta U}{m_e}}$$

To figure out the initial potential energy one must consider each <u>pair</u> of charges.

$$U_i = U_{12i} + U_{13i} + U_{14i} + U_{23i} + U_{24i} + U_{34i}$$

Before getting too crazy and computing all these, notice we are going to do the subtraction $\Delta U = U_f - U_i$. Any terms which remain unchanged between the initial and final state will drop out! Since only charge 4 moves we expect only terms with a 4 in the subscripts will survive the subtraction. Furthermore, the changes associated with 2 and 4 should be the same as the changes associated with 3 and 4. Therefore we may write:

$$\Delta U = (U_{14f} - U_{14i}) + (U_{24f} - U_{24i}) + (U_{34f} - U_{34i})$$
$$\Delta U = (U_{14f} - U_{14i}) + 2(U_{34f} - U_{34i})$$

<u>Initial</u> distances between charges are:

$$r_{14i} = \sqrt{2}a \qquad r_{24i} = a \qquad r_{34i} = a$$

<u>Final</u> distances between charges are:

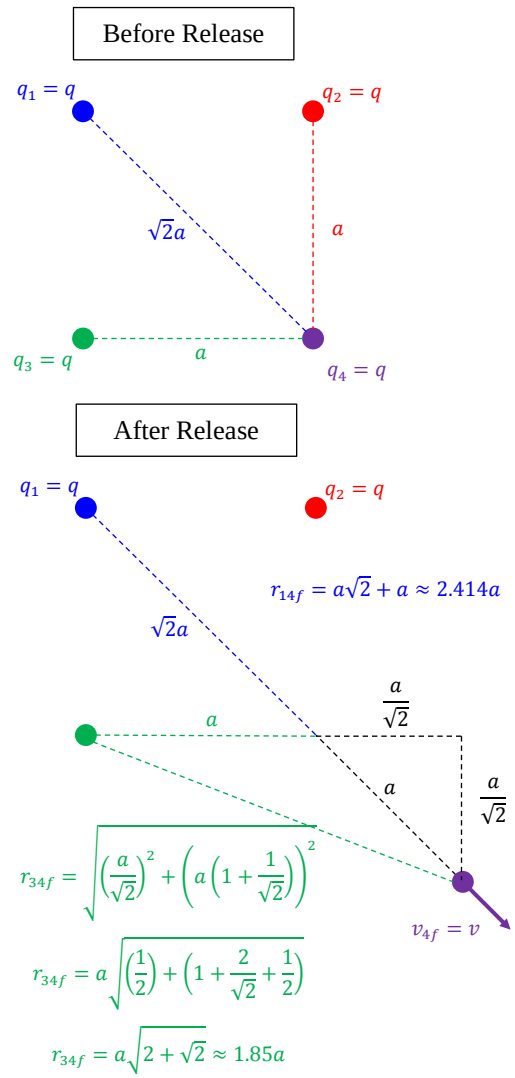
$$r_{14f} = 2.414a \quad r_{24f} = 1.848a \quad r_{34f} = 1.848a$$

See figure at right to see why…

Since every term has $\frac{kq^2}{a}$ I factor that out from the get go.

$$\Delta U = \frac{kq^2}{a}\left\{\left(\frac{1}{2.414} - \frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{1.848} - 1\right)\right\}$$
$$\Delta U = -1.211\frac{kq^2}{a}$$

This negative sign makes sense. Charge 4 moves away reducing the concentration of positive charge. This lowers the potential of the charges to explode apart (lowers potential energy).



$$r_{34f} = \sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + \left(a\left(1 + \frac{1}{\sqrt{2}}\right)\right)^2}$$

$$r_{34f} = a\sqrt{\left(\frac{1}{2}\right) + \left(1 + \frac{2}{\sqrt{2}} + \frac{1}{2}\right)}$$

$$r_{34f} = a\sqrt{2 + \sqrt{2}} \approx 1.85a$$

Note: the above radical cannot be denested. I guess one could call it "un-denestable"? These are the things I choose to do with my time…

Anyways...we can shove this into

$$v_f = \sqrt{-\frac{2\Delta U}{m}} \approx 1.56\sqrt{\frac{kq^2}{ma}}$$

By the way, did you notice this is NOT the answer to part d of that first square problem? To make this the answer to part d of that first problem charge 4 traveled distance $\sqrt{2}a$…not a. Also, you can't simply multiply the denominator inside the root $\sqrt{2}$ to make the answers match. The components do not all change by the same factor.

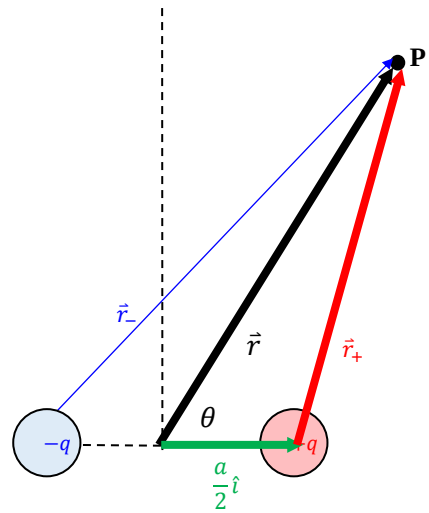
PARTY ON!

24.13

a) First notice that the point **P** is located at the coordinate (x, y). That means we could write the vector $\vec{r}$ in one of two ways:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$
$$\vec{r} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$$

where θ is the angle shown in the figure. This will be useful later.



Look at those three bold vectors carefully to notice the graphical vector addition. Many times physicists rewrite an off-origin vector using graphical vector addition.

$$\vec{r} = \frac{a}{2}\hat{\imath} + \vec{r}_+$$

Tail to tip in the HOUSE!

Upon rearranging one finds

$$\vec{r}_+ = \vec{r} - \frac{a}{2}\hat{\imath}$$

$$\vec{r}_+ = (x\hat{\imath} + y\hat{\jmath}) - \frac{a}{2}\hat{\imath}$$

$$\boldsymbol{\vec{r}_+ = \left(x - \frac{a}{2}\right)\hat{\imath} + y\hat{\jmath}}$$

Similarly one finds

$$\boldsymbol{\vec{r}_- = \left(x + \frac{a}{2}\right)\hat{\imath} + y\hat{\jmath}}$$

Now do the same ol' crap.

$$V_{tot} = V_- + V_+$$

$$V_{tot} = \frac{k(-q)}{r_-} + \frac{k(+q)}{r_+}$$

$$V_{tot} = -\frac{kq}{\sqrt{\left(x + \frac{a}{2}\right)^2 + y^2}} + \frac{kq}{\sqrt{\left(x - \frac{a}{2}\right)^2 + y^2}}$$

$$V_{tot} = kq\left(\frac{-1}{\sqrt{\left(x + \frac{a}{2}\right)^2 + y^2}} + \frac{1}{\sqrt{\left(x - \frac{a}{2}\right)^2 + y^2}}\right)$$

Some people might use $x = r\cos\theta$ and $y = r\sin\theta$ at this point. I choose to hold off another line or 2. Notice that ugly denominator. I spy with my little eye there will be an $x^2 + y^2$ if I square out that $\left(x + \frac{a}{2}\right)^2$. This is nice because it cleans up using $r^2 = x^2 + y^2$. Similar crap happens in the other radical.

$$V_{tot} = kq\left(\frac{-1}{\sqrt{x^2 + ax + \frac{a^2}{4} + y^2}} + \frac{1}{\sqrt{x^2 - ax + \frac{a^2}{4} + y^2}}\right)$$

$$V_{tot} = kq\left(\frac{-1}{\sqrt{r^2 + ax + \frac{a^2}{4}}} + \frac{1}{\sqrt{r^2 - ax + \frac{a^2}{4}}}\right)$$

More next page…

Now rewrite using $x = r\cos\theta$

$$V_{tot} = kq\left(\frac{-1}{\sqrt{r^2 + ar\cos\theta + \frac{a^2}{4}}} + \frac{1}{\sqrt{r^2 - ar\cos\theta + \frac{a^2}{4}}}\right)$$

This technically answers what I asked for.
If you are really sick you might notice you could've just used the law of cosines to get r_+ & r_-!!!
Everyone once in a while that thing is actually useful! It is useful pretty much whenever you have two vectors adding to make a third and they don't form a right triangle.

Let's get *nasty* now…
Typically we are concerned with *molecules* with a certain *dipole moment*. If you don't know what that is, hold on a few minutes and I'll tell you. Point being, we are interested in how *molecules* interact with each other. That is why we might care about writing this equation in terms of r (the distance from the molecule) and θ (the angle relative to the molecule's dipole moment). Most importantly, for most molecules, the distance a is tiny! It is totally reasonable to assume $r \gg a$ for many practical situations (unless two molecules are right on top of each other).

If using molecules and ($r \gg a$), V_{tot}cleans up nicely. Basically, get rid of all the $\frac{a^2}{4}$'s with impunity!!!!

WATCH OUT PUNK! You might think you can rid yourself of any terms with a in it but we can do better than that with the binomial approximation. How does one know this? By experience. Basically, you learn quickly if you eliminate *all* terms with a the answer is $V_{tot} = 0$. While this is a good *first* approximation, we want to see the *next order term* in the approximation so we now only eliminate the $\frac{a^2}{4}$ crap. Hope this makes sense…

$$V_{tot} = kq\left(\frac{-1}{\sqrt{r^2 + ar\cos\theta}} + \frac{1}{\sqrt{r^2 - ar\cos\theta}}\right)$$

Now get binomial expansion on it. If $r \gg a$ let $\delta = \frac{a}{r} \ll 1$. To get this involved factor out r^2 from each radical!

$$V_{tot} = kq\left(\frac{-1}{r\sqrt{1 + \frac{a}{r}\cos\theta}} + \frac{1}{r\sqrt{1 - \frac{a}{r}\cos\theta}}\right)$$

$$V_{tot} = \frac{kq}{r}\left(-\left(1 + \frac{a}{r}\cos\theta\right)^{-1/2} + \left(1 - \frac{a}{r}\cos\theta\right)^{-1/2}\right)$$

$$V_{tot} \approx \frac{kq}{r}\left(-\left(1 + \left(-\frac{1}{2}\right)\frac{a}{r}\cos\theta + \cdots\right) + \left(1 - \left(-\frac{1}{2}\right)\frac{a}{r}\cos\theta + \cdots\right)\right)$$

$$V_{tot} \approx \frac{kq}{r}\left(-\left(1 - \frac{a}{2r}\cos\theta + \cdots\right) + \left(1 + \frac{a}{2r}\cos\theta + \cdots\right)\right)$$

$$V_{tot} \approx \frac{k\boldsymbol{qa}\cos\theta}{r^2}$$

That term in hot pink is the magnitude of the dipole moment. If you look online, you'll see this derivation is different than most but the final result is the same. The math ends up the same if you use dot products ($r_+ = \sqrt{{r_+}^2} = \sqrt{\vec{r}_+ \cdot \vec{r}_+}$). Follow it through to see those $\cos\theta$ terms popping out of the dot products. This can be a cool math trick to keep in your quiver…

It turns out a great deal of crap can be calculated in physics and chemistry using dipole moments. Try a web search for "electric dipole moment" after this problem if you haven't barfed from boredom yet.

24.14

a) Equipotentials is the name given to a curve or surface in space containing points at equal potential.

b) The innermost equipotential is at 10.0 V, the middle equipotential is at 4.00 V, and the outer equipotential is at 2.00 V.

c) Using the approximation we find

$$E_r \approx \frac{|\Delta V|}{\Delta r}$$

Here $\Delta r = 5.0\text{ cm} - 2.5\text{ cm}$ is the distance between the 2.00 V and 4.00 V equipotentials..

$$E_r \approx \frac{|4.00\text{ V} - 2.00\text{ V}|}{2.5\text{ cm}}$$

$$E_r \approx \frac{2.00\text{ V}}{0.025\text{ m}}$$

$$\boldsymbol{E_r \approx 80\frac{\mathbf{V}}{\mathbf{m}}}$$

d) Using the other equation gives

$$E = \frac{kq}{r^2}$$

Here $r = 0.0375\text{ m}$ is distance from source (the point charge) to the point of interest **P**.

$$E = 71\frac{\text{N}}{\text{C}} = 71\frac{\text{V}}{\text{m}}$$

e) The % difference is

$$\%\Delta E = \frac{E_{approx} - E}{E} \times 100\%$$

$$\%\Delta E = \frac{80 - 71}{71} \times 100\%$$

$$\%\Delta E = 13\%$$

Not great, but decent. Using equipotentials closer together would get much better results.

24.15

a) The radii for the equipotentials are tabulate at right.

V (V)	r (m)
-5.0	1.798
-10.0	0.899
-15.0	0.599
-20.0	0.450
-25.0	0.360

b) As long as you have chosen equipotentials separated by equal potential differences, closer spacing implies larger electric field. Think: if $E_r \approx \frac{|\Delta V|}{\Delta r}$ and $|\Delta V|$ is constant between any two equipotentials, smaller Δr gives bigger numbers for E_r.

c) From previous chapters, know the electric field points radially inwards towards a negative charge. Notice the electric field points perpendicular to equipotentials and from high to low potential (towards lower voltage).

d) The estimate gives

$$E_r \approx \frac{|\Delta V|}{\Delta r} \approx \frac{|(-15.0\text{V}) - (-10.0\text{V})|}{0.899\text{ m} - 0.599\text{ m}} \approx 16.67\frac{\text{V}}{\text{m}}$$

Halfway between the -15.0 V & -10.0 V equipotentials is about $r = \frac{0.899+0.599}{2} \approx 0.749\text{ m}$

Using the other formula gives

$$E = \frac{kq}{r^2} = 16.02\frac{\text{V}}{\text{m}}$$

The % difference is

$$\%\Delta E = \frac{E_{approx} - E}{E} \times 100\%$$

$$\%\Delta E = 4\%$$

Not bad at all…

24.16

a) When equipotentials are closely spaced it implies large electric fields.
b) Large fields imply large forces and, in turn, large accelerations.
c) A positive charge move in the same direction as the electric field. Electric fields point from high voltage to low voltage (towards lower potential).A positive charge moving in same direction as field would thus tend to move towards lower potential.
d) *Negative* charges move *opposite* the electric field (*negative* charges move towards *higher* potential).

24.17 Use a simulation.

24.18

According to the figure drawn the distance from source (dq) to point of interest is $r = x$ and we find

$$dV = \frac{k\,dq}{r}$$

$$dV = \frac{k\,(\lambda\,dx)}{x}$$

The electric potential at **A** is thus

$$V = \int_i^f dV$$

$$V = \int_a^{L+a} \frac{k\,(\lambda\,dx)}{x}$$

For a uniformly charged rod

$$\lambda = \frac{Q_{total}}{Length_{total}}$$

$$\lambda = \frac{Q}{L} \quad in\ this\ case$$

Notice the density is a constant and it will pull out of the integral. Therefore we find

$$V = \int_a^{L+a} \frac{k\,(\lambda\,dx)}{x}$$

$$V = k\lambda \int_a^{L+a} \frac{dx}{x}$$

$$\boldsymbol{V = k\lambda \left(\ln x\right)_a^{L+a}}$$

$$V = k\lambda(\ln|L+a| - \ln|a|)$$

$$V = k\lambda \ln\frac{|L+a|}{|a|}$$

$$\boldsymbol{V = k\lambda \ln\left|\frac{L+a}{a}\right| = \frac{kQ}{L}\ln\left|\frac{L+a}{a}\right|}$$

Note: I usually skip straight from the first bold equation to the second. I recommend memorizing this shortcut. To me this final bold result makes more sense as the term inside the logarithm now has no units.

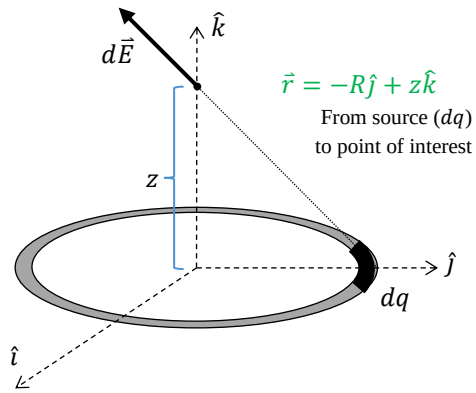
What checks can we do on our answer?

- The units on the final expression look good (k is on top, units of $\frac{\mathrm{C}}{\mathrm{m}}$ on the rest of it just like $\frac{kq}{r}$ for a point charge).
- As $a \to 0$ our point of interest would be very close to the charged rod. The potential blows up to infinity when $a \to 0$ so this seems reasonable.
- As $a \to \infty$ we would be very far from the rod. We suspect the potential should drop to zero. As $a \to \infty$ one sees $V \approx k\lambda \ln\left|\frac{a}{a}\right| = 0$. This also makes sense.

24.19

Going from the source to the point of interest gives

$$\vec{r} = -R\hat{\jmath} + z\hat{k}$$
$$r = \sqrt{R^2 + z^2}$$
$$dV = \frac{k\ dq}{\sqrt{R^2 + z^2}}$$



A small arc of charge has length $ds = R\ d\theta$.
The small arc has charge $dq = \lambda\ ds = \lambda\ R\ d\theta$.

The electric potential at **A** is thus

$$V = \int_i^f dV$$
$$V = \int_0^{2\pi} \frac{k\ (\lambda\ R\ d\theta)}{\sqrt{R^2 + z^2}}$$

Note: be sure to use radians (not degrees) for the limits of your angles.

For a uniformly charged rod

$$\lambda = \frac{Q_{total}}{Length_{total}}$$
$$\lambda = \frac{Q}{2\pi R} \quad in\ this\ case$$

Notice the density is a constant and it will pull out of the integral. Also notice R and z are both constants!!!
If you go to a different portion of charge on the ring $\vec{r}$ changes but r *doesn't* change. See the subtle difference? Therefore we find

$$V = \int_0^{2\pi} \frac{k\ (\lambda\ R\ d\theta)}{\sqrt{R^2 + z^2}}$$
$$V = \frac{k\lambda R}{\sqrt{R^2 + z^2}} \int_0^{2\pi} d\theta$$
$$V = \frac{k\lambda R}{\sqrt{R^2 + z^2}}\ \theta \Big|_0^{2\pi}$$
$$V = \frac{k\lambda R}{\sqrt{R^2 + z^2}} (2\pi - 0)$$
$$V = \frac{2\pi k\lambda R}{\sqrt{R^2 + z^2}}$$
$$V = \frac{2\pi k \left(\frac{Q}{2\pi R}\right) R}{\sqrt{R^2 + z^2}}$$
$$\boldsymbol{V = \frac{kQ}{\sqrt{R^2 + z^2}}}$$

What checks can we do on our answer?

- The units look good (k is on top, units of $\frac{\text{C}}{\text{m}}$ on the rest of it just like $\frac{kq}{r}$ for a point charge).
- When $z = 0$ our point of interest is equidistant from all charge on the rod. The potential *should be* $V = \frac{kQ}{R}$ and the final result for electric potential matches this expectation.
- As $z \gg R$ we would be very far from the ring. The radius of the ring is tiny from this great distance. Our result *should be* identical to being distance z from a point charge ($V = \frac{kQ}{z}$). This also checks out with our result.

24.20 Build up the disk out of rings.

$$V_{disk} = \int_i^f dV_{ring}$$

From previous problem we know a ring of radius R and total charge Q has

$$V = \frac{kQ}{\sqrt{R^2 + z^2}}$$

Now we want to build a disk of radius R and charge Q out of little rings of radius r and charge dq.

$$dV_{ring} = \frac{k\ dq}{\sqrt{r^2 + z^2}}$$

But wait! We know this is a 2D object…use a 2D density.

$$\sigma = \frac{Q}{A} = \frac{Q}{\pi R^2}$$

The area of my building ring is length times thickness. Length is $2\pi r$ and thickness is dr. This implies

$$dq = \sigma\ dA = 2\pi r\ \sigma\ dr$$

$$V_{disk} = \int_i^f \frac{k\ (2\pi r\ \sigma\ dr)}{\sqrt{r^2 + z^2}}$$

The smallest ring used to build the disk has radius $r_i = 0$ while the largest has $r_f = R$.

$$V_{disk} = 2\pi k\sigma \int_0^R \frac{r\ dr}{\sqrt{r^2 + z^2}}$$

From there do a u-sub and finish it out. I'm confident you can find a final answer on the internet.

Going further: If a washer was used the obvious change is the limits of integration.
WATCH OUT! The charge density will also change as Q spreads out over a different area!!!

24.21

a) As θ increases the charge density increases. We expect the ends of the arc have lots of charge while the middle of the arc (directly above the origin) has no charge!

b) $[\alpha] = \frac{[\lambda]}{[\theta^2]} = \frac{\mathrm{C/m}}{no\ units} = \frac{\mathrm{C}}{\mathrm{m}}$

c) Total charge relates to density using

$$Q = \int_i^f dq$$

$$Q = \int_i^f \lambda\, ds$$

$$Q = \int_i^f \lambda\, R\, d\theta$$

$$Q = \int_i^f \alpha\theta^2\, R\, d\theta$$

$$Q = \alpha R \int_i^f \theta^2\, d\theta$$

$$Q = \alpha R \int_{-\pi/4}^{+\pi/4} \theta^2\, d\theta$$

$$Q = 2\alpha R \int_0^{+\pi/4} \theta^2\, d\theta$$

$$Q = 2\alpha R \left[\frac{\theta^3}{3}\right]_0^{+\pi/4}$$

$$\boldsymbol{Q = \frac{\alpha\pi^3 R}{96}}$$

d) We usually build arcs out of point charges

$$V = \int_i^f \frac{k\, dq}{r}$$

In this special case our point of interest is at the origin. Each dq is the same distance $r = R$ from source to the point of interest. Notice this is distance is a constant and pulls out of the integral!

$$V = \int_i^f \frac{k\, dq}{R}$$

$$V = \frac{k}{R}\int_i^f dq$$

$$V = \frac{kQ}{R}$$

$$\boldsymbol{V = \frac{k\alpha\pi^3}{96}}$$

Notice the R's cancel and the units check (k on top time Coulombs over meters).

e) If an odd power of θ was used, the arc would carry equal amounts of positive and negative charge. The symmetry of the situation produces zero electric potential at the origin.

24.22 We usually build rods out of point charges

$$V = \int_i^f \frac{k\,dq}{r}$$

Here $r = \sqrt{x^2 + L^2}$ and $dq = \lambda\,dx$ where, because rod is uniform, use $\lambda = \frac{Q}{L}$.

$$V = \int_i^f \frac{k\,\lambda\,dx}{r}$$

$$V = \frac{kQ}{L}\int_i^f \frac{dx}{(x^2 + L^2)^{1/2}}$$

$$V = \frac{kQ}{L}\int_0^L \frac{dx}{(x^2 + L^2)^{1/2}}$$

Grab a table and go nuts…

$$V = \frac{kQ}{L}\ln\left|x + \sqrt{x^2 + L^2}\right|_0^L$$

$$V = \frac{kQ}{L}\ln\left(\frac{L + \sqrt{2}L}{L}\right)$$

$$V = \frac{kQ}{L}\ln\left(1 + \sqrt{2}\right)$$

$$\boldsymbol{V \approx 0.881\frac{kQ}{L}}$$

Think: if a non-uniform density of $\lambda = \alpha x$ is used the integral is easier (can do with u-sub).

24.23 We usually build rods out of point charges

$$V = \int_i^f \frac{k\,dq}{r}$$

Here $r = \sqrt{x^2 + L^2}$ and $dq = \lambda\,dx$ where, because rod is uniform, use $\lambda = \frac{Q}{3L}$.

$$V = \int_i^f \frac{k\,\lambda\,dx}{r}$$

$$V = \frac{kQ}{3L}\int_i^f \frac{dx}{(x^2 + L^2)^{1/2}}$$

$$V = \frac{kQ}{3L}\int_{-3L/2}^{3L/2} \frac{dx}{(x^2 + L^2)^{1/2}}$$

Grab a table and go nuts…

$$V = \frac{kQ}{3L}\ln\left|x + \sqrt{x^2 + L^2}\right|_{-\frac{3}{2}L}^{+\frac{3}{2}L}$$

$$V = \frac{kQ}{3L}\ln\left(\frac{+\frac{3}{2}L + \frac{\sqrt{13}}{2}L}{-\frac{3}{2}L + \frac{\sqrt{13}}{2}L}\right)$$

$$V = \frac{kQ}{3L}\ln\left(\frac{+3 + \sqrt{13}}{-3 + \sqrt{13}}\right)$$

$$\boldsymbol{V \approx 0.797\frac{kQ}{L}}$$

Think: if a non-uniform density of $\lambda = \alpha x$ the rod would carry equal amounts of positive and negative charge. The symmetry of the situation produces zero electric potential at **P**. No integral required. Be careful…in general you do not get zero potential with equal amounts of + & -…they also must be equidistant from the point of interest.

24.24

a) If we want zero electric potential at the origin the arc and point charge must have opposite signs. We expect the charge density on the arc must be negative (opposite the positive point charge).

b) You might think calculus is required to learn about the arc but it is not. Because all parts of the arc are equidistance from the point of interest (the origin) we find

$$V = \int_i^f \frac{k\,dq}{r}$$

$$V = \int_i^f \frac{k\,dq}{R}$$

$$V = \frac{k}{R}\int_i^f dq$$

$$V_{arc} = \frac{kQ_{arc}}{R}$$

Because the arc carries uniform charge distribution we may use $Q_{arc} = \lambda s$ where s is the length of the arc.

$$s = R(2 \times 60°\ converted\ to\ radians)$$

$$s = \frac{2\pi R}{3}$$

Therefore $Q_{arc} = \frac{2\pi\lambda R}{3}$.

We are told there is zero potential at the origin. Therefore

$$V_{arc} + V_{point} = 0$$

$$V_{arc} = -V_{point}$$

$$\frac{kQ_{arc}}{R} = -\frac{kQ_{point}}{R}$$

$$Q_{arc} = -Q_{point}$$

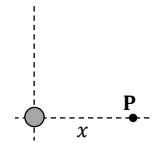
$$\frac{2\pi\lambda R}{3} = -Q$$

$$\boldsymbol{\lambda = -\frac{3Q}{2\pi R}}$$

Notice the charge density is negative as expected from part a. Units check.

24.25 Calculating electric field from electric potential

Point charge q at origin. Point of interest **P** on x-axis distance x from charge.



$$V = \frac{kq}{x}$$
$$\vec{E} = -\vec{\nabla}V$$

That weird symbol $\vec{\nabla}$ is called the "del operator" (also called "nabla"). When used on a scalar, such as V, we say you are taking the gradient. In 2D Cartesian form

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}$$

Those slightly italicized derivatives are called partial derivatives. They are actually easier to use than total derivatives. I will explain how they work in this and the next example.

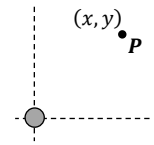
$$\vec{E} = -\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}\right)V$$
$$\vec{E} = -\left(\frac{\partial}{\partial x}V\hat{\imath} + \frac{\partial}{\partial y}V\hat{\jmath}\right)$$
$$\vec{E} = -\frac{\partial}{\partial x}\left(\frac{kq}{x}\right)\hat{\imath} - \frac{\partial}{\partial y}\left(\frac{kq}{x}\right)\hat{\jmath}$$

Here is how that partial derivative works. A partial with respect to x means treat everything except x in the equation as a constant. A partial with respect to y means treat everything except y in the equation as a constant.

$$\vec{E} = -kq\frac{\partial}{\partial x}\left(\frac{1}{x}\right)\hat{\imath} - \frac{kq}{x}\frac{\partial}{\partial y}(1)\hat{\jmath}$$
$$\vec{E} = -kq\left(-\frac{1}{x^2}\right)\hat{\imath} + 0\hat{\jmath}$$
$$\vec{E} = \frac{kq}{x^2}\hat{\imath} + 0\hat{\jmath}$$

Go back and look at the picture. Verify that is the correct electric field at point **P**.

24.26 Calculating electric field from electric potential



Point charge q at origin. Point of interest **P** at the coordinate (x, y).

$$V = \frac{kq}{\sqrt{x^2 + y^2}} = \frac{kq}{(x^2 + y^2)^{1/2}} = kq(x^2 + y^2)^{-1/2}$$

$$\vec{E} = -\vec{\nabla} V$$

$$\vec{E} = -\left(\frac{\partial}{\partial x} V\hat{\imath} + \frac{\partial}{\partial y} V\hat{\jmath}\right)$$

$$\vec{E} = -\frac{\partial}{\partial x}\left(kq(x^2 + y^2)^{-1/2}\right)\hat{\imath} - \frac{\partial}{\partial y}\left(kq(x^2 + y^2)^{-1/2}\right)\hat{\jmath}$$

$$\vec{E} = -kq\frac{\partial}{\partial x}\left((x^2 + y^2)^{-1/2}\right)\hat{\imath} - kq\frac{\partial}{\partial y}\left((x^2 + y^2)^{-1/2}\right)\hat{\jmath}$$

$$\vec{E} = -kq\left(\left(-\frac{1}{2}\right)(x^2 + y^2)^{-3/2}(2x)\right)\hat{\imath} - kq\left(\left(-\frac{1}{2}\right)(x^2 + y^2)^{-3/2}(2y)\right)\hat{\jmath}$$

Notice each term has a 2 to cancel the ½'s from the derivatives. Also notice each term has $(x^2 + y^2)^{-3/2}$ which can be factored out. Lastly, notice each ½ from the derivatives has a minus sign to cancel the red minus sign out front!

$$\vec{E} = +kq(x^2 + y^2)^{-3/2}(x\hat{\imath} + y\hat{\jmath})$$

Now use the fact that in polar coordinates $r = (x^2 + y^2)^{1/2}$ and move that ugly crap to the basement to get

$$\vec{E} = +\frac{kq}{r^3}(x\hat{\imath} + y\hat{\jmath})$$

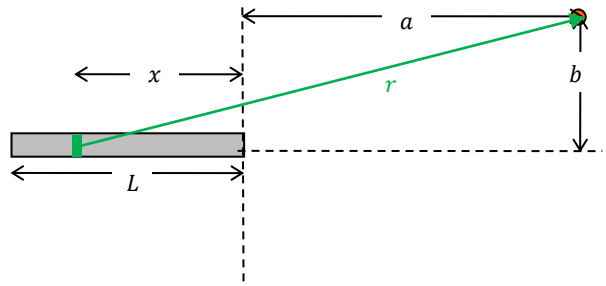
Go back and look at the picture. Verify that is the correct electric field at point **P**.

What is the point of this again?

It is pretty easy to write down electric potential compared to electric field.
Computers can pretty much take any derivative you can imagine and do it symbolically.

If you want to find the electric field of some crazy assembly of charge you could first find the electric potential at some arbitrary point with the coordinate (x, y, z). Then take the 3D gradient of the electric potential and you know the electric field. Make a computer do all the heavy lifting and you'll seem like genius! From that point, use a computer model to predict how something moves or what have you and make a better toaster.

24.27 One finds



$$\vec{r} = (x+a)\hat{\imath} + b\hat{\jmath}$$
$$r = \sqrt{(x+a)^2 + b^2}$$
$$dV = \frac{k\,dq}{\sqrt{(x+a)^2 + b^2}}$$

For a uniformly charged rod

$$\lambda = \frac{Q_{total}}{Length_{total}}$$
$$\lambda = \frac{Q}{L} \quad in\ this\ case$$

Notice the density is a constant and it will pull out of the integral. Therefore we find

$$V = \int_0^L \frac{k\,(\lambda\,dx)}{\sqrt{(x+a)^2 + b^2}}$$
$$V = k\lambda \int_0^L \frac{dx}{\sqrt{(x+a)^2 + b^2}}$$

I assumed x is *distance* (not *position*) from the origin. I must assume x only takes on *positive* values. My limits run from 0 to $+L$. If I had chosen to let x be *position* the limits would be 0 to $-L$ but would use $r = \sqrt{(a-x)^2 + b^2}$

Now let $u = x + a$ which gives $du = dx$.
Notice if you plug in $x = 0$ into the u equation you find $u = a$.
If you plug in $x = L$ you find $u = L + a$.

$$V = k\lambda \int_a^{L+a} \frac{du}{\sqrt{u^2 + b^2}}$$

From there you are off to the races. Let's not bother to finish this one now. It is better just to use it as a practice for setting up a problem. **Move on to the next thing now.**

Parting shot: think to yourself how you could check this result against the previous rod problem when point A was on the x-axis instead of up in the first quadrant…
Said another way, what value of b should make the equations match up?

Big picture:
Usually in science and engineering applications we care about the forces acting on objects and their motion. Coulomb's law will get this done but many real-world calculations can be messy.

Some problems can be solved using electrical potential energy. For others, it can be easier to first calculate an electric field. In general, the nastier a problem looks, the harder it is to calculate electric fields. For super nasty problems, it is probably easier to first calculate the electric potential since it has no vector crap in it. Once one knows an algebraic expression for electric potential, you could shove that equation into a computer and have it take a numerical gradient (take a derivative numerically…similar to calculating slopes using trendlines). This gives you information about an electric field and thus the electric force acting on a particle and can help you understand the motion of a real world thing.

Finally, the motion of electrons in circuits relates directly to electric potential very easily. It is usually much easier to discuss circuits in terms of electric potential (V) than in terms of electric fields ($\vec{E}$).

For these reasons, we got more and more abstract, going from force to field and ultimately electric potential.
Now we get to use electric potential to party all day long (do a significant number of problems with less effort)!

24.28

a) We expect the electric field points upwards (positive z-direction).

b) In general we know

$$\vec{E} = -\left(\frac{\partial}{\partial x}V\hat{\imath} + \frac{\partial}{\partial y}V\hat{\jmath} + \frac{\partial}{\partial z}V\hat{k}\right)$$

Since only the z-component is expected to survive, we may use

$$\vec{E} = -\frac{\partial}{\partial z}V\hat{k}$$

$$\vec{E} = -\frac{\partial}{\partial z}\left(\frac{kQ}{\sqrt{R^2 + z^2}}\right)\hat{k}$$

From a previous chapter the answer is

$$\vec{E}_{ring} = \frac{kQz}{(R^2 + z^2)^{3/2}}\hat{k}$$

24.29 You already know all the answers. Do it if you are having fun with these.

24.30

a) We know $E_x = -\frac{dV}{dx} = -(\text{slope of } V \text{ vs. } x \text{ plot})$. If the slope is zero we know electric field is zero. This occurs between $0 \rightarrow 200$ nm, $400 \rightarrow 500$ nm and $900 \rightarrow 1000$ nm.

b) We know $E_x = -(\text{slope of } V \text{ vs. } x \text{ plot})$. A *negative slope* gives a *positive* E_x. This occurs for $x = 500 \rightarrow 900$ nm.

c) Potential energy is given by

$$U = qV = (+e)(20 \text{ mV}) = 20 \text{ meV} = 20 \times 10^{-3} \text{ eV} = 3.2\underline{0}4 \times 10^{-21} \text{ J}$$

d) Energy is conserved. In equation form you can writes that as initials equal the finals

$$\boldsymbol{K_i + U_i = K_f + U_f}$$

or you can write it as change in kinetic equals negative change in potential

$$\boldsymbol{\Delta K = -\Delta U}$$

Verify you are comfortable with both statements before moving on.

If a positive charge goes to a location of higher potential (more positive voltage) the potential energy increases. On the graph you can think of this as protons going uphill gain potential energy. Using the bold red equation, we see a proton going uphill gains potential energy while losing kinetic energy. The proton must go up and over the 80 mV hill to reach the 40 mV pleateau at $x = 1000$ nm.

$$\Delta K = -\Delta U$$

$$K_f - K_i = -\Delta U$$

The problem asked about the *minimum* initial speed for the proton to reach $x = 1000$ nm

Assume the speed at the top *of the highest* hill to cross is essentially zero. This tells us $K_f = 0$.

$$-K_i = -\Delta U$$

$$K_i = \Delta U$$

$$\frac{1}{2} m_p v_i^2 = q(\Delta V)$$

$$\frac{1}{2} m_p v_i^2 = (+e)(\Delta V)$$

WATCH OUT!!! Here ΔV is not V. The change in voltage going up the hill is $\Delta V = V_f - V_i = 60$ mV!!!

Solving for initial speed gives

$$v_i = \sqrt{\frac{2e\Delta V}{m_p}} = \sqrt{\frac{2(1.602 \times 10^{-19} \text{ C})(60 \times 10^{-3} \text{ V})}{(1.67 \times 10^{-27} \text{ kg})}} = 3.3\underline{9}3 \frac{\text{km}}{\text{s}}$$

24.31

a) WATCH OUT! *Protons* released from rest move *with* the electric field while *electrons* released from rest move *against* the electric field. The slope at $x = 300$ nm is positive. Since $E_x = -(slope)$ the electric field points to the left. The electron released from rest will thus move to the right!
TIP: protons tend to go to lower voltages, electrons tend to go to higher voltages (unless they have non-zero initial speed).
TIP: protons tend to go downhill…electrons tend to go uphill!

b) Recall acceleration relates to force using $\vec{a} = \frac{\vec{F}}{m}$. Force relates to field using $\vec{F} = q\vec{E}$.

$$a_x = \frac{F_x}{m_e}$$
$$a_x = \frac{qE_x}{m_e}$$
$$a_x = \frac{(-e)(-slope)}{m_e}$$

Notice minus signs cancelled!

$$a_x = \frac{e\left(\frac{rise}{run}\right)}{m_e}$$
$$a_x = \frac{(1.602 \times 10^{-19}\ \text{C})\left(\frac{60\ \text{mV}}{200\ \text{nm}}\right)}{(9.11 \times 10^{-31}\ \text{kg})}$$

Think: slope is the same for any section between 200 & 400 nm. I could've also used $slope = \frac{30\ \text{mV}}{100\ \text{nm}}$.

$$\boldsymbol{a_x = 5.2\underline{7}6 \times 10^{16}\,\frac{\text{m}}{\text{s}^2}}$$

If you are wondering about the units, the equation $U = qV$ tells us $1\ \text{C·V} = 1\ \text{J} = 1\ \text{N·m} = 1\ \frac{\text{kg·m}^2}{\text{s}^2}$.

c) Released from rest, the electron starts with zero kinetic energy and $U = qV = (-e)(50\ \text{mV}) = -50\ \text{meV}$.
As the negative charge moves to the right it goes to higher *potential* but lower *potential energy*.

$$\Delta U = q\Delta V$$
$$\Delta U = (-e)\Delta V$$
$$\Delta U = -e\Delta V$$

<u>For negative charges</u> we see increasing voltage will lower potential energy!
As the electron goes to the top of the hill is converts potential energy to kinetic.
The potential energy at the peak is $U = qV = (-e)(80\ \text{mV}) = -80\ \text{meV}$.
The electron *lost* 30 meV of potential energy and gained 30 meV of kinetic energy.
The *negative charge* reaches max speed *at the top* of the hill.
After crossing the peak it comes back down.
At $x = 800$ nm the electron is back $U = qV = (-e)(50\ \text{mV}) = -50\ \text{meV}$.
At $x = 800$ nm the 30 meV of kinetic energy has all been converted back to potential.
At $x = 800$ nm the electron has no kinetic energy and is at rest.
The electron reverses direction at $x = 800$ nm.
Notice the slope there is negative.
This implies the electric field is positive ($E_x = -slope$).
The electron at rest experiences a force directed *opposite* the electric field.
The electron does go back to the left!
THE ELECTRON OSCILLATES BETWEEN $x = 300$ nm & $x = 800$ nm!!!

More on next page…

d) Now use

$$\Delta K = -\Delta U$$
$$K_f - K_i = -\Delta U$$

Because the electron was released from rest we know $K_i = 0$.

$$K_f = -\Delta U$$

WATCH OUT! This time the minus sign does <u>not</u> cancel at this step!!!

$$\frac{1}{2} m_e v_f^2 = -(-e)(\Delta V)$$

Once we plug in the charge of the electron (not charge *magnitude*) we see the minus sign <u>does</u> cancel!
WATCH OUT!!! Here ΔV is not V.
The change in voltage going up the hill is $\Delta V = 80\text{ mV} - 50\text{ mV} = 30\text{ mV}$!!!

$$v_f = \sqrt{\frac{2e\Delta V}{m_e}} = 10\underline{2}.7\,\frac{\text{km}}{\text{s}}$$

Notice electron speeds tend to be much higher than proton speeds.
The proton mass is about 1800 times greater than the electron mass.

24.32

a) At $x = -10.0$ cm we see the slope is negative. If we want a charge to move to the right we imagine we want something that will roll downhill. If released from rest we know positive charges roll downhill & negative charges roll uphill. Choose a positive charge if you want it to go to the right.

b) The charge starts from rest (all potential energy) at voltage $V = 6.00$ μV.

$$Energy_{total} = K_i + U_i$$
$$Energy_{total} = 0 + q(6.00\ \mu\text{V})$$
$$Energy_{total} = q(6.00\ \mu\text{V})$$

Because this system conserves energy, it can never exceed this total energy.
While the charge moves to the right from $x = -10.0$ cm to $x = 0$ cm it speeds up.
It continues to move to the right at constant speed between $0 \rightarrow 10$ cm.
It then continues to move to the right but slows down until reaching at $x = 25$ cm.
It continues at a slow constant rate over the plateau between $25 \rightarrow 30$ cm.
It speeds up moving from 30 cm → 35 cm.
Travels at constant speed between 35 cm → 40 cm.
It then slows down until all kinetic energy converts back to potential energy.
This occurs when the *potential* is once again 6.00 μV at about $x \approx 44$ cm.
It then reverses direction and goes back to the starting position of $x = -10.0$ cm.
A positive charge released from rest @ $x = -10$ cm oscillates between $x = -10$ cm & $x = 44$ cm.

c) We know

$$\Delta K = -\Delta U$$
$$\Delta U = -\Delta K$$
$$q\Delta V = -\left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right)$$
$$q\Delta V = -\frac{1}{2}m\left(v_f^2 - v_i^2\right)$$

WATCH OUT!!! Students often screw this up. Recall $\left(v_f^2 - v_i^2\right) \neq \left(v_f - v_i\right)^2$.
If you don't believe me, use the FOIL method on the right side and you'll see it right away.

$$q\Delta V = \frac{1}{2}m\left(v_i^2 - v_f^2\right)$$

WATCH OUT! I got rid of the minus sign by flipping the order of initial and final inside the parentheses.

$$q = \frac{m}{2\Delta V}\left(v_i^2 - v_f^2\right)$$
$$q = \frac{m}{2\Delta V}\left(v_i^2 - v_f^2\right)$$
$$q = \frac{2.00 \times 10^{-22}}{2(10.00\ \mu\text{V} - 6.00\ \mu\text{V})}\left(\left(285\frac{\text{m}}{\text{s}}\right)^2 - \left(80.0\frac{\text{m}}{\text{s}}\right)^2\right)$$
$$q = 1.871 \times 10^{-12}\ \text{C}$$

I assumed three sig figs.
In real life it is unlikely to get that much precision on those blue numbers from reading a graph.
I asked you to answer in the form $q = (some\ \#)e$.

$$q = (1.871 \times 10^{-12}\ \text{C})\frac{e}{e}$$
$$q = \left(\frac{1.871 \times 10^{-12}\ \text{C}}{e}\right)e$$
$$q = \left(\frac{1.871 \times 10^{-12}\ \text{C}}{1.602 \times 10^{-19}}\right)e$$
$$\boldsymbol{q = 1.168 \times 10^{7}\ e}$$
$$\boldsymbol{q = 11.68 \times 10^{6}\ e}$$

***q* is equivalent to the charge on about 12 million protons**

24.33

a) We are dealing with a negative charge. Positive charges tend to flow downhill but negative charges tend to flow uphill. The electron will have the most trouble crossing the valley between $x = 35 \rightarrow 40$ cm.
The electron needs enough initial kinetic energy to pass through the valley between $x = 35 \rightarrow 40$ cm.

$$\Delta K = -\Delta U$$
$$K_f - K_i = -\Delta U$$

The problem asked about the *minimum* initial speed.
Assume the final speed *at the deepest valley* is zero.

$$-K_i = -\Delta U$$
$$K_i = \Delta U$$
$$\frac{1}{2}m_e v_i^2 = q(\Delta V)$$
$$\frac{1}{2}m_e v_i^2 = (-e)(\Delta V)$$
$$\frac{1}{2}m_e v_i^2 = -e(\Delta V)$$
$$v_i = \sqrt{\frac{-2e\Delta V}{m_e}}$$

WATCH OUT! That minus sign on the right hand side is supposed to be there. It will work out…
The change in voltage going to the bottom of the deepest valley is

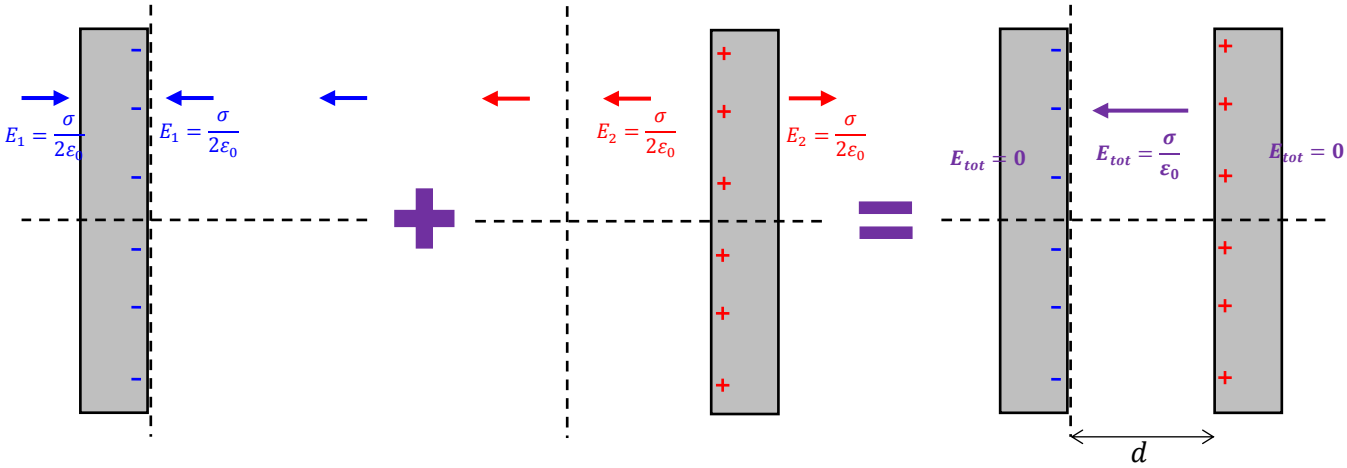
$$\Delta V = V_f - V_i = (-6.00\ \mu\text{V}) - (10.00\ \mu\text{V}) = -16.00\ \mu\text{V}$$

$$\boldsymbol{v_i = 2.3\underline{7}2\,\frac{\text{km}}{\text{s}}}$$

b) This was a total trick question. The electron will never reverse direction!!!
It starts out at $x = 50$ cm moving to the left.
As it travels down to the bottom of the first valley it moves to lower voltage.
Negative charges moving to lower voltage gain potential energy and lose kinetic.
The electron slows down travelling from $x = 45$ cm to $x = 40$ cm.
The electron will now continue at very slow speed from $x = 40$ cm to $x = 35$ cm.
We know moving very slowly because we were told we slightly more than the minimum energy required to reach $x = 5$ cm.
From $x = 35$ cm to $x = 30$ cm the electron is moving to higher voltage and speeds up.
From $x = 30$ cm to $x = 25$ cm the electron travels at constant speed (no slope, no E, no force, no accel).
From $x = 25$ cm to $x = 10$ cm the electron is moving to lower voltage and slows down.
WATCH OUT! This valley is not as deep as the first one!
From $x = 25$ cm to $x = 10$ cm the electron converts some, but not all, kinetic energy to potential energy.
From $x = 10$ cm to $x = 0$ cm the electron travels at constant speed (no slope, no E, no force, no accel).
From $x = 0$ cm to $x = -20$ cm the electron is moving to higher voltage and speeds up.
At this point the electron is gone from our system and we don't know what will happen.

24.34

a) The electric field is probably best determined using the pictures below.



The mathematical trick we are using here is called the superposition principle. The idea is this: get the electric field of one plate all by itself. Get the electric field of the other plate all by itself. The total electric field *vector* is the sum of the two individual electric field *vectors*.

Notice $E_1 = \frac{\sigma}{2\varepsilon_0}$ is the *magnitude* of the electric field caused by the left plate where σ is a *positive* number. The arrows pointing *towards* the negative plate accounts for the left plate being negative.

Final Note: the superposition principle is also frequently used while analyzing waves. Try a web search for "Fourier synthesis" for kicks.

b) Do the integral

$$\Delta V = -\int_i^f \vec{E} \cdot d\vec{s}$$

$$\Delta V = -\int_0^d \left(\frac{\sigma}{\varepsilon_0}(-\hat{\imath})\right) \cdot dx\hat{\imath}$$

$$\Delta V = \frac{\sigma}{\varepsilon_0}\int_0^d dx$$

$$\Delta V = \frac{\sigma}{\varepsilon_0} d$$

Notice, in this special case, $\Delta V > 0$ so what we have found is the magnitude of the electric potential difference. In some cases you might need to take the absolute value to get rid of a minus sign in order to determine the magnitude of the electric potential difference.

c) Notice the part in purple is the magnitude of the electric field.

$$\Delta V = Ed$$

In class I like to say it this way: "Vee equals Ed." This equation is useful. Memorize it. Anytime you have parallel plates it tends to show up. Why so useful? In a lab we tend to connect power supplies to parallel plate capacitors. The voltage from the power supply is ΔV. Plate spacing d is often known. We use this set-up in a lab to create a constant electric field of magnitude E. Remember the field points from positive plate to negative plate (or from high voltage to low).

More next page…

d) I thought this is rather interesting. Notice the *static* electric field is zero inside the conductors themselves (in two chapters we get beyond static fields). For now restrict ourselves to the region between the plates. Let us integrate from the origin to some arbitrary position x between the plates

$$\Delta V_{0\ to\ x} = -\int_0^x \left(\frac{\boldsymbol{\sigma}}{\boldsymbol{\varepsilon_0}}(-\hat{\imath})\right) \cdot dx\hat{\imath}$$

$$\Delta V_{0\ to\ x} = \frac{\sigma}{\varepsilon_0} x$$

If we let the negative plate be ground (0 Volts) we know the voltage will linearly increase going towards the positive plate.

$$\Delta V_{0\ to\ x} = V(x) - 0\text{ Volts} = \frac{\sigma}{\varepsilon_0} x$$

$$V(x) = \frac{\sigma}{\varepsilon_0} x$$

Note: when we reach the other plate $x = d$ and the voltage is $V_{\substack{right \\ plate}} = \frac{\sigma}{\varepsilon_0} d$.

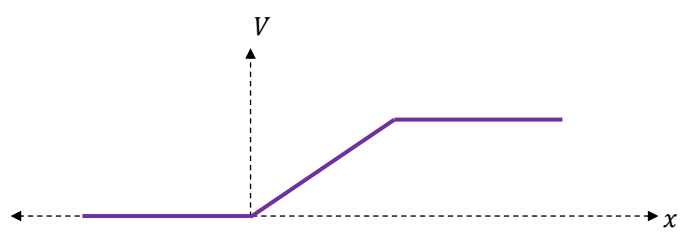
Now go from the right plate to some other point even farther right. In this region the electric field is zero.

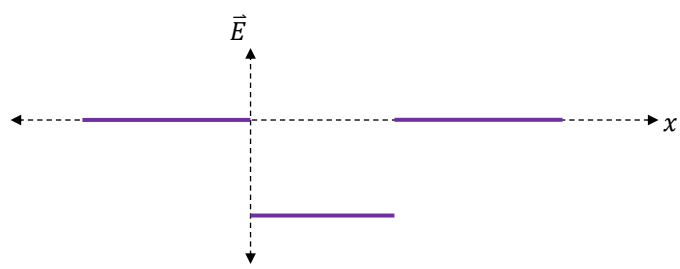
$$\Delta V_{d\ to\ somewhere\ to\ the\ right} = -\int_d^{right\ of\ d} (\mathbf{0}\hat{\imath}) \cdot dx\hat{\imath}$$

$$\Delta V_{d\ to\ somewhere\ to\ the\ right} = 0$$

PAY ATTENTION! The voltage is <u>not</u> zero out there…there is no <u>*change*</u> in voltage as you go past the right plate. Similarly, the voltage on the left side of the left plate should be the same as the voltage at the left plate. In summary we know the following. Notice plate thickness t is not important in parallel plates.

	$x < 0$	$0 < x < d$	$x > d$
Potential	$\mathbf{0}$	$\frac{\sigma}{\varepsilon_0} x$	$\frac{\sigma}{\varepsilon_0} d$
Field (+ = "to the right")	$\mathbf{0}$	$-\frac{\sigma}{\varepsilon_0}$	$\mathbf{0}$





Electrical engineers and solid state physicists (condensed matter) look sharp.
Think about a marble on the V versus x plot: it would roll at constant speed on the flat sections then have downhill acceleration on the slope. Positive charges flow like this in the circuit. Electrons would do the opposite (uphill acceleration).

24.35

a) We are outside the sphere at all times, use $\vec{E} = \frac{kQ}{r^2}\hat{r}$

$$\Delta V_{C\to\infty} = -\int_i^f \vec{E}\cdot d\vec{s}$$

$$V_\infty - V_C = -\int_i^f \left(\frac{kQ}{r^2}\hat{r}\right)\cdot(dr\,\hat{r})$$

Recall, for a point charge $V_\infty = 0$. Note: this is *not* true for infinite sheets or lines of charge…
The minus signs on each side will cancel. Also, the dot product inside gives $\hat{r}\cdot\hat{r} = 1$.

$$V_C = \int_{r_C}^{\infty} \frac{kQ}{r^2}dr$$

$$V_C = -\frac{kQ}{r}\bigg|_{r_C}^{\infty}$$

$$V_C = \frac{kQ}{r_C}$$

The result is what we have already been using for some time.
Notice we could've chosen any point outside the sphere for point C. This means the result for V_C is the potential at any radius outside the sphere.

b) At point B use $r = R$! Therefore $V_B = \frac{kQ}{R}$.

c) We are *inside* the sphere at all times, use $\vec{E} = \frac{kQr}{R^3}\hat{r}$

$$\Delta V_{A\to B} = -\int_i^f \vec{E}\cdot d\vec{s}$$

$$V_B - V_A = -\int_i^f \left(\frac{kQr}{R^3}\hat{r}\right)\cdot(dr\,\hat{r})$$

$$-V_A = -V_B - \int_i^f \left(\frac{kQr}{R^3}\hat{r}\right)\cdot(dr\,\hat{r})$$

$$V_A = V_B + \int_i^f \frac{kQr}{R^3}dr$$

$$V_A = \frac{kQ}{R} + \frac{kQ}{R^3}\int_{r_A}^{R} r\,dr$$

$$V_A = \frac{kQ}{R} + \frac{kQ}{R^3}\cdot\frac{r^2}{2}\bigg|_{r_A}^{R}$$

$$V_A = \frac{kQ}{R} + \frac{kQ}{R^3}\left(\frac{R^2}{2} - \frac{r_A^2}{2}\right)$$

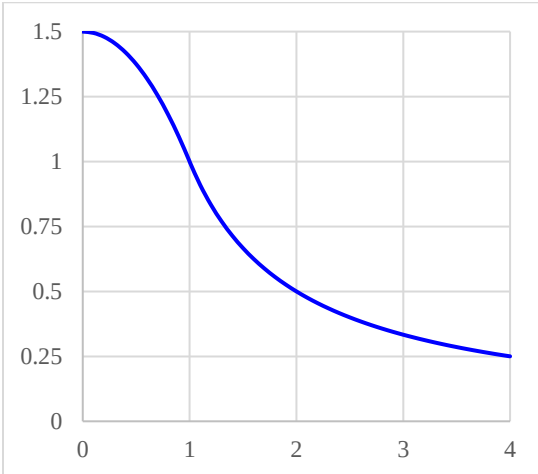
$$V_A = \frac{kQ}{R} + \frac{kQ}{2R} - \frac{kQ}{R^3}\frac{r_A^2}{2}$$

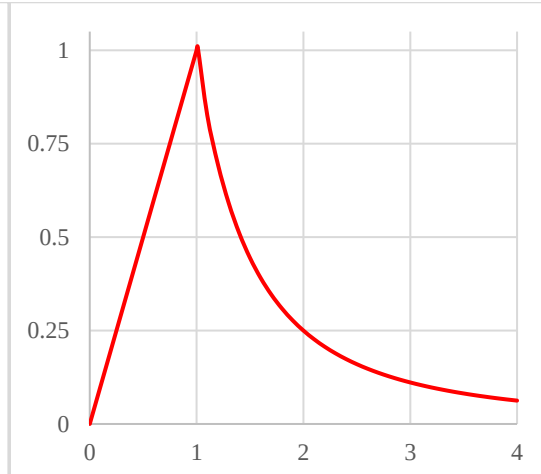
$$V_A = \frac{3kQ}{2R} - \frac{kQ}{R^3}\frac{r_A^2}{2}$$

$$\boldsymbol{V = \frac{kQ}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right)}$$

Plot on next page…

d) Outside we know $V = \frac{kQ}{r}$ and inside we know $V = \frac{kQ}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right)$. Notice the two functions match at the special radius $r = R$. Using $k = Q = R = 1$ and ignoring units gives the plot shown below. Vertical axis for blue plot is is potnetial, vertical axis for red plot is electric field magnitude, horiztonal axis is r for both..





e) If a conductor was used instead of an insulator, we know there is zero *static* eletric field inside. We will get to current flow in about two chapters.
If electric field inside is zero, consider how part c above changes:

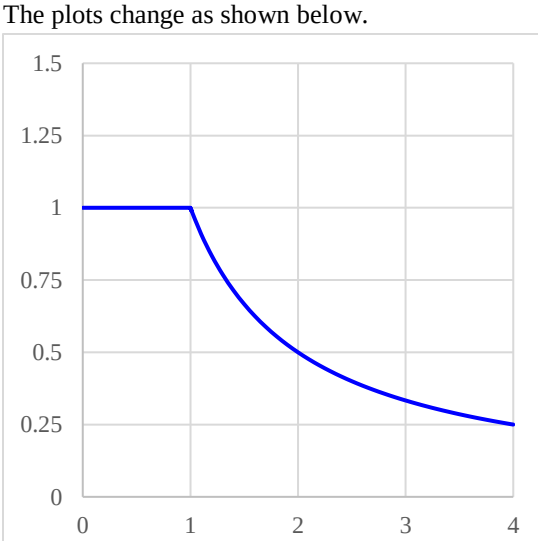
$$\Delta V_{A\to B} = -\int_i^f \vec{E} \cdot d\vec{s}$$

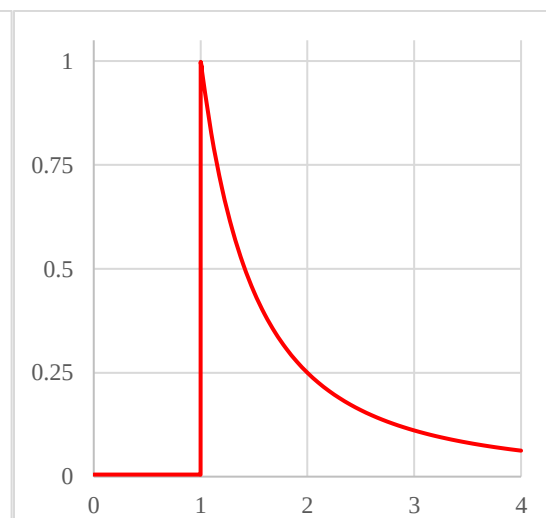
$$V_B - V_A = -\int_i^f 0 \cdot (dr\,\hat{r})$$

$$V_B - V_A = 0$$

$$V_B = V_A$$

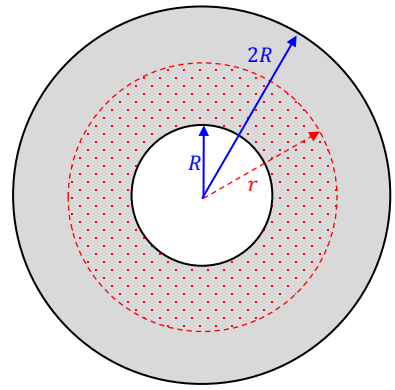
Inside the conductor the electric potential is zero (assuming *static* electric field).
The plots change as shown below.





24.36 First I do Gauss's law to determine the electric fields. Double check the two fields match at the special radii $r = R$ and $r = 2R$. Looks good. Also, the units looks good. See tab le below.

$\boldsymbol{E_{R<r<2R}}$
Cross-section of sphere



$E_{r<R}$	0	No charge enclosed, no electric field!
$E_{R<r<2R}$	$\frac{kQ}{r^2}\cdot\frac{(r^3-R^3)}{7R^3}$	$E(4\pi r^2) = \frac{q_{in}}{\varepsilon_0}$ $E(4\pi r^2) = \frac{Q}{\varepsilon_0}\frac{V_{enclosed}}{V_{total}}$ $E(4\pi r^2) = \frac{Q}{\varepsilon_0}\frac{\frac{4}{3}\pi(r^3-R^3)}{\frac{4}{3}\pi((2R)^3-R^3)}$ $E(4\pi r^2) = \frac{Q}{\varepsilon_0}\frac{(r^3-R^3)}{7R^3}$ $E = \frac{Q}{(4\pi r^2)\varepsilon_0}\frac{(r^3-R^3)}{7R^3}$
$E_{r>2R}$	$\frac{kQ}{r^2}$	<u>Outside</u> object with spherical geometry $\vec{E}$ is same as point charge (use $Q = sum\ of\ all\ q's\ inside$)

Outside the sphere the problem is nearly identical our previous work in parts **24.35**.

$$\boldsymbol{V_{r>2R} = \frac{kQ}{r}}$$

$$\boldsymbol{V_{2R} = \frac{kQ}{2R}}$$

WATCH OUT! The outer radius of this sphere is $2R$.
We get that extra 2 in the basement of V at the shell's surface!

For some arbitrary r in the region $R < r < 2R$:

$$\Delta V_{r\ to\ 2R} = -\int_r^{2R}\left(\frac{kQ}{\tilde{r}^2}\cdot\frac{(\tilde{r}^3-R^3)}{7R^3}\hat{\tilde{r}}\right)\cdot\left(d\tilde{r}\,\hat{\tilde{r}}\right)$$

$$\Delta V_{r\ to\ 2R} = -\int_r^{2R}\left(\frac{kQ\tilde{r}}{7R^3}-\frac{kQ}{7\tilde{r}^2}\right)d\tilde{r}$$

$$\Delta V_{r\ to\ 2R} = -\frac{kQ}{7}\int_r^{2R}\left(\frac{\tilde{r}}{R^3}-\frac{1}{\tilde{r}^2}\right)d\tilde{r}$$

$$\Delta V_{r\ to\ 2R} = -\frac{kQ}{7}\left[\frac{\tilde{r}^2}{2R^3}+\frac{1}{\tilde{r}}\right]_r^{2R}$$

$$\Delta V_{r\ to\ 2R} = -\frac{kQ}{7}\left\{\left(\frac{(2R)^2}{2R^3}+\frac{1}{2R}\right)-\left(\frac{(r)^2}{2R^3}+\frac{1}{r}\right)\right\}$$

$$\Delta V_{r\ to\ 2R} = -\frac{kQ}{7}\left\{\frac{5}{2R}-\frac{r^2}{2R^3}-\frac{1}{r}\right\}$$

Continues next page…

From previous page we found:

$$\Delta V_{r\ to\ 2R} = -\frac{kQ}{7}\left\{\frac{5}{2R} - \frac{r^2}{2R^3} - \frac{1}{r}\right\}$$

$$V_{2R} - V_{r<2R} = -\frac{kQ}{7}\left\{\frac{5}{2R} - \frac{r^2}{2R^3} - \frac{1}{r}\right\}$$

$$\boldsymbol{\frac{kQ}{2R}} - V_{r<2R} = -\frac{kQ}{7}\left\{\frac{5}{2R} - \frac{r^2}{2R^3} - \frac{1}{r}\right\}$$

Flipping all the signs cleans up a lot.

$$-\boldsymbol{\frac{kQ}{2R}} + V_{r<2R} = \frac{kQ}{7}\left\{\frac{5}{2R} - \frac{r^2}{2R^3} - \frac{1}{r}\right\}$$

Also, distribute on the right side to get:

$$-\boldsymbol{\frac{kQ}{2R}} + V_{r<2R} = \boldsymbol{\frac{5kQ}{14R}} - \frac{kQr^2}{14R^3} - \frac{kQ}{7r}$$

Group the two bold terms and reduce fraction to find

$$V_{r<2R} = \boldsymbol{\frac{12kQ}{14R}} - \frac{kQr^2}{14R^3} - \frac{kQ}{7r}$$

Factor out

$$\boldsymbol{V_{r<2R} = \frac{kQ}{7R}\left(6 - \frac{r^2}{2R^2} - \frac{R}{r}\right)}$$

Before going on, the units look good on each term. If I plug in $r = 2R$ I get

$$V_{r=2R} = \frac{kQ}{7R}\left(6 - \frac{(2R)^2}{2R^2} - \frac{R}{(2R)}\right)$$

$$V_{r=2R} = \frac{kQ}{7R}\left(6 - 2 - \frac{1}{2}\right)$$

$$V_{r=2R} = \frac{kQ}{7R}\left(\frac{7}{2}\right)$$

$$\boldsymbol{V_{r=2R} = \frac{kQ}{2R}}$$

Notice this matches what we got from the result outside the sphere. Looks promising.
At the inner surface of the shell ($r = R$) we find

$$V_{r=R} = \frac{kQ}{7R}\left(6 - \frac{(R)^2}{2R^2} - \frac{R}{(R)}\right)$$

$$V_{r=R} = \frac{kQ}{7R}\left(6 - \frac{1}{2} - 1\right)$$

$$\boldsymbol{V_{r=R} = \frac{9kQ}{14R}}$$

Finally, we know $E_{r<R} = 0$. This implies the is no *change* in voltage as we go from $r = R$ to $r < R$.

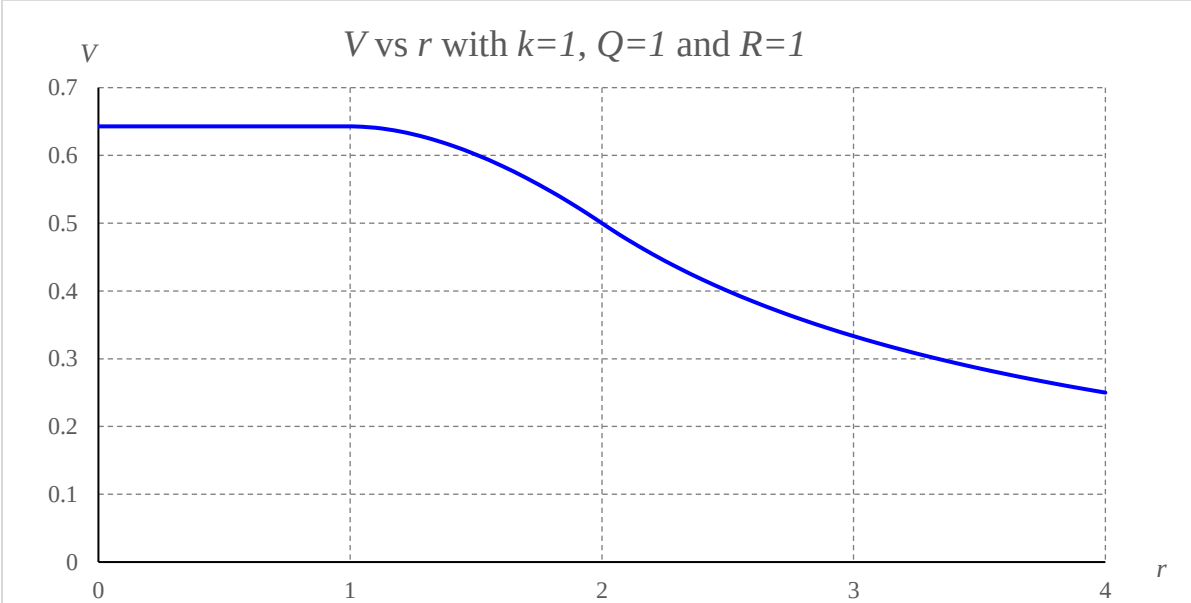
$$\boldsymbol{V_{r<R} = \frac{9kQ}{14R}}$$

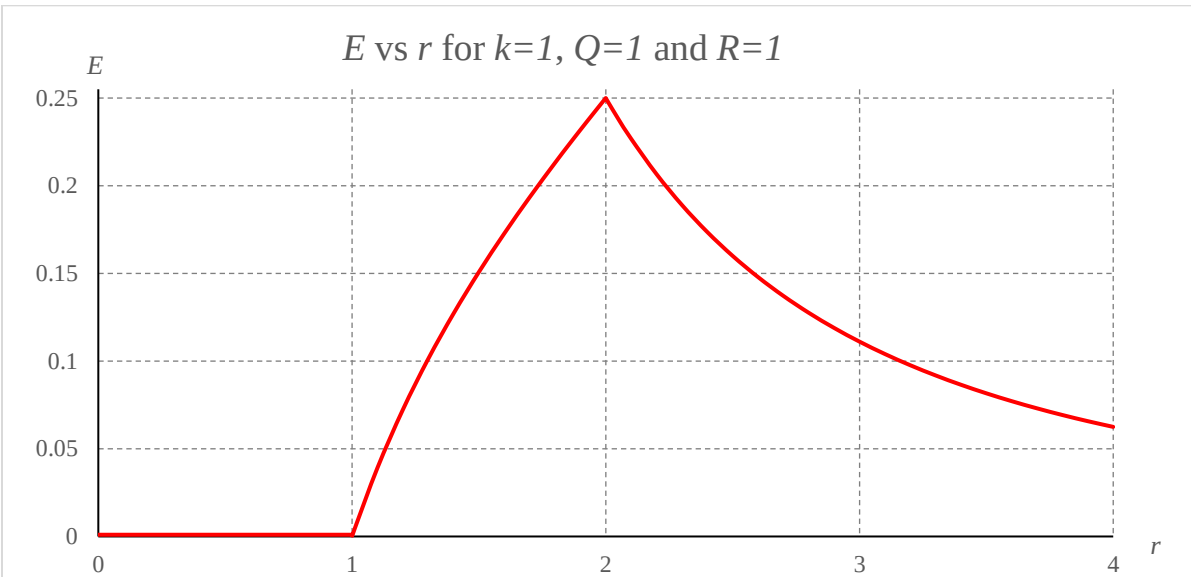
Tabulating results

$V_{r<R}$	$\frac{9kQ}{14R}$
$V_{R<r<2R}$	$\frac{kQ}{7R}\left(6 - \frac{r^2}{2R^2} - \frac{R}{r}\right)$
$V_{r>2R}$	$\frac{kQ}{r}$

Plots on the next page…

Here are the plots of V vs r and E vs r. As usual, I set $k = 1$, $Q = 1$ and $R = 1$.





24.37 The arc geometry is tougher because, in this class, we have only determined the electric field at the center of arcs. The equation we have been using in the last few problems is

$$\Delta V = -\int_i^f \vec{E} \cdot d\vec{s}$$

To use this formula we would first need to figure out the electric field caused by an arc at an arbitrary point in space. Without that equation for $\vec{E}$ we can't turn the crank and do the math.

If you had an arc symmetric about the vertical axis (with similarly symmetric charge distribution) I could imagine a case where you could determine the electric field for any point on the vertical axis. From there you could do integrations between points on the vertical axis only. By symmetry the electric field would point along the vertical axis and would make the integrals a lot more doable.

24.38

a) The units of a are $\frac{\mathrm{N \cdot m}^n}{\mathrm{C}}$ while the units of b are $\frac{\mathrm{N}}{\mathrm{C \cdot m}^m}$.

b) This is probably what most students would do:

$$\Delta V = -\int_i^f \vec{E} \cdot d\vec{s}$$

$$\Delta V_{\mathbf{B}\ to\ \mathbf{A}} = -\int_{x=c}^{x=d} \left(-\frac{a}{x^n} + bx^m\right)\hat{\imath} \cdot dx\hat{\imath}$$

$$\Delta V_{\mathbf{B}\ to\ \mathbf{A}} = \int_c^d (ax^{-n} - bx^m)dx$$

$$\Delta V_{\mathbf{B}\ to\ \mathbf{A}} = \left(\frac{ax^{-n+1}}{-n+1} - \frac{bx^{m+1}}{m+1}\right)_c^d$$

WATCH OUT! The problems asked for $\Delta V_{\mathbf{A}\ to\ \mathbf{B}}$ but we did **B** to **A** to reduce minus sign errors. Notice: in the last term below on the right hand side I flipped the limits to account for the minus sign…

$$\Delta V_{\mathbf{A}\ to\ \mathbf{B}} = -\Delta V_{\mathbf{B}\ to\ \mathbf{A}} = -\left(\frac{ax^{-n+1}}{-n+1} - \frac{bx^{m+1}}{m+1}\right)_c^d = \left(\frac{ax^{-n+1}}{-n+1} - \frac{bx^{m+1}}{m+1}\right)_d^c$$

$$\Delta V_{\mathbf{A}\ to\ \mathbf{B}} = \left(\frac{a}{(-n+1)c^{n-1}} - \frac{bc^{m+1}}{m+1}\right) - \left(\frac{a}{(-n+1)d^{n-1}} - \frac{bd^{m+1}}{m+1}\right)$$

Ugly..but done. The interesting stuff comes next (fingers crossed).

c) $\Delta V_{\mathbf{B}\ to\ \mathbf{A}} = -\Delta V_{\mathbf{A}\ to\ \mathbf{B}}$…just flip the sign on the previous result.

d) If $n = 1$ or $m = -1$ it means you can't use the power rule. You'd have natural logs popping up instead of our previous result. Strictly speaking, the previous result is not correct unless you specific we are excluding these particular integers.

e) The electron will speed up going from **A** to **B** if the electric field points from **B** to **A**. Think about it, electrons feel a force *opposite* the direction of the electric field. We could similarly think about the motion of the electron motion in terms of electric *potential*. Positive charges speed up as they go towards lower voltage, negative charges speed up as they go towards positive voltage. If $\Delta V_{\mathbf{A}\ to\ \mathbf{B}} > 0$ the electron should speed up. Note: if you are worried the electron is gaining both kinetic and potential energy that is <u>not</u> the case. The electron gains potential energy $\Delta U = (-e)\Delta V_{\mathbf{A}\ to\ \mathbf{B}}$. Notice it *gains* a *negative* amount of potential energy…in other words, it loses *potential energy* even though it increases its electrical *potential*. See that subtle difference in wording?

24.39 With a sphere or spherical shell there is a finite amount of charge. With an infinite line of charge, no matter how far away you are, the potential can never fall to zero in the presence of infinite charge. It turns out we can still use our formulas to get changes in potential near infinite wires. Fortunately, this is all that is physically meaningful in most problems.

24.40 You could first determine the electric field from a finite line of charge as was done in Ch 22. Then you could integrate in the same way. For a finite line charge, you could once again assume the electric potential at infinity is zero.

24.41

a) The electric potential at the surface of an isolated sphere is $\frac{kQ}{R}$. Therefore we find the spheres are at potentials $\frac{kq}{a}$ and $-\frac{kq}{b}$.

b) The potential difference (might as well take high voltage minus low voltage) is

$$\Delta V_{lo\ to\ hi} = V_{hi} - V_{lo}$$

$$\Delta V_{lo\ to\ hi} = \frac{kq}{a} - \left(-\frac{kq}{b}\right)$$

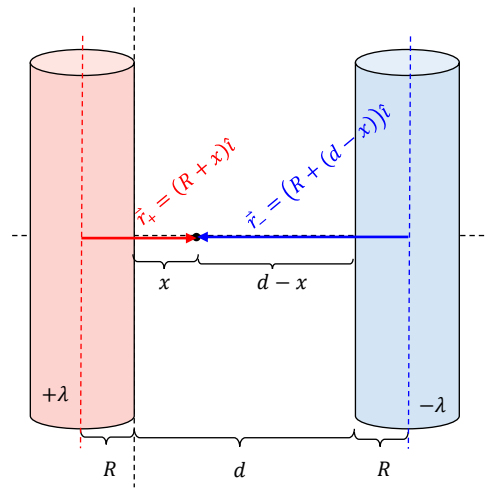
$$\Delta V_{lo\ to\ hi} = kq\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$\Delta V_{lo\ to\ hi} = \frac{kq(a+b)}{ab}$$

c) No difference if we had used insulators as long as charge is uniformly distributed.

d) If the charges are polarized, our derivation of $V_{r>R}$ is flawed. Why? Because the perfect symmetry is broken and our Gauss's law derivation of $\vec{E}$ fails. Think about it: if polarized the electric field vectors aren't always perfectly aligned radially outwards. $\oint \vec{E} \cdot d\vec{A} \neq EA$. More complicated methods are required.

24.42 Let us use the superposition principle to find the net electric field vector between the lines of charge. I will zoom in a bit to make the vectors easier to understand. See figure at right.



We want to determine the net electric field vector at some arbitrary point between the two cylinders. We may as well do this on the x-axis.

The electric field vector from the line charge on the left is

$$\vec{E}_+ = \frac{2k\lambda}{r_+}\hat{\imath}$$

$$\vec{E}_+ = \frac{2k\lambda}{R+x}\hat{\imath}$$

The electric field vector from the line charge on the left is

$$\vec{E}_- = \frac{2k\lambda}{r_-}\hat{\imath}$$

$$\vec{E}_- = \frac{2k\lambda}{R+d-x}\hat{\imath}$$

Notice this field points to the *right* ($+\hat{\imath}$) and the above equation will reflect that for all $x < d$.
The total electric field vector is thus

$$\vec{E}_{total} = \vec{E}_+ + \vec{E}_-$$

$$\vec{E}_{total} = \frac{2k\lambda}{R+x}\hat{\imath} + \frac{2k\lambda}{R+d-x}\hat{\imath}$$

$$\vec{E}_{total} = 2k\lambda\left(\frac{1}{R+x} + \frac{1}{R+d-x}\right)\hat{\imath}$$

Notice you could now shove this electric field into

$$\Delta V = -\int_i^f \vec{E}\cdot d\vec{s}$$

$$\Delta V_{\mathbf{x=0}\ to\ \mathbf{x=d}} = -\int_{x=0}^{x=d} 2k\lambda\left(\frac{1}{R+x} + \frac{1}{R+d-x}\right)\hat{\imath}\cdot dx\hat{\imath}$$

Since we are going from a plus charge to a minus charge we expect the result should be $\Delta V_{\mathbf{x=0}\ to\ \mathbf{x=d}} < 0$.

Notice the integral would explode if we hadn't properly accounted for the radius of each line charge…
In case you are wondering, we know the change in electric potential must be the same for any two paths because the electric force is conservative. This is useful because it allowed us to integrate along the x-axis and make the vectors slightly easier to think about. That said, you could integrated along any other path between the two cylinder surfaces and obatin the same result.

Note: if you redid the integral with limits of 0 to x you could plot V vs x and $\vec{E}$ vs x (at least for the region between the two cylinders). Outside you might need to revisit the electric field equation as you expect partial cancellation in the regions to the left of both cylinders or to the right of both cylinders. Also note, the electric field equation is only valid in the plane defined by the two cylinders in the limit they are both infinitely large ($x \ll L$).

24.43a

A screenshot of my code output is shown below.

Apologies for me being lazy with formatting…

If you'd like help, talk to me in office hours.

You should find the electric field is less than $1\,\frac{\mathrm{V}}{\mathrm{m}}$ for distances less than about 4 m from the origin.